FOLDY-WOUTHUYSEN TRANSFORMATION FOR RELATIVISTIC PARTICLES IN EXTERNAL FIELDS

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1 INTRODUCTION

The Foldy-Wouthuysen (FW) representation occupies a special place in the quantum theory. For relativistic particles in external fields, operators in this representation have the same form as in the nonrelativistic quantum theory. The relations between the operators in the FW representation are similar to those between the respective classical quantities. For example, the position operator is $\boldsymbol{r}$ and the momentum one is $\boldsymbol{p} = -i\hbar\nabla$ even for relativistic particles in external fields. Only the FW representation possesses these properties considerably simplifying the transition to the semiclassical description. As a result, this representation provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum mechanics.

There are three levels of description of interaction of relativistic particles with strong external fields:

Classical physics

Relativistic quantum mechanics

Quantum field theory

The solution of problems on every level is necessary.

There is a limited class of Hamiltonians those transformation to the FW representation is exact:

A. G. Nikitin, J. Phys. A **31**, 3297 (1998).

A. J. Silenko, J. Math. Phys. **44**, 2952 (2003).

In the general case, the exact FW transformation has been performed by Eriksen:

E. Eriksen, Phys. Rev. **111**, 1011 (1958).

However, it is difficult to use the derived FW Hamiltonian for obtaining a semiclassical limit of the Dirac equation for relativistic particles because the Hamiltonian is very cumbersome and contains roots of Dirac matrix operators. The methods of performing the FW transformation and finding the semiclassical limit of wave equations for relativistic spin-1/2 and spin-1 particles have been developed in Refs.

A. J. Silenko, J. Math. Phys. **44**, 2952 (2003)

and

A. J. Silenko, Zs. Eksp. Teor. Fiz. **123**, 883 (2003) [JETP **96**, 775 (2003)],

respectively. These methods can be mostly used in the weak-field approximation.

The new method of the FW transformation for relativistic particles of arbitrary spin in strong external fields is proposed. This method is based on the above mentioned developments. However, it does not need any definite commutation relations between even and odd operators. The general form of a transition operator has been found. The final FW Hamiltonian can be expanded into a power series in the Planck constant. Since just this constant defines the order of magnitude of quantum corrections, the transition to the semiclassical approximation becomes trivial. As an example, interactions of scalar particles and Dirac ones with a strong electromagnetic field have been analyzed. The equation of spin motion of Dirac particles in the strong electromagnetic field has been derived.

We use the designations $[\ldots,\ldots]$ and $\{\ldots,\ldots\}$ for commutators and anticommutators, respectively.

2 FOLDY-WOUTHUYSEN METHOD AND PREVIOUSLY OBTAINED RESULTS

In this section, we review known methods of the FW transformation and some of previously obtained results.

In the classical work by Foldy and Wouthuysen

L. L. Foldy, S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950),

two different transformations, for free relativistic particles and for nonrelativistic particles in electromagnetic fields have been carried out. In the general case, transformation to a new representation described by the wave function Ψ' is performed with the unitary operator U:

$$\Psi' = U\Psi = e^{iS}\Psi,$$

where $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ is the wave function (bispinor) in the Dirac representation.

The Hamiltonian can be split into operators commuting and noncommuting with the operator β:

$$\mathcal{H} = \beta m + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta. \tag{1}$$

The Hamiltonian $\mathcal{H}$ is Hermitian. We assume that both operators

$\mathcal{E}$ and $\mathcal{O}$ are also Hermitian.

For nonrelativistic particles in an electromagnetic field, the FW transformation can be performed with the operator

J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964)

$$S = -\frac{i}{2m}\beta\mathcal{O}. \tag{2}$$

The transformed Hamiltonian can be written in the form

$$\begin{array}{c}\mathcal{H}' = \mathcal{H} + i[S,\mathcal{H}] + \frac{i^2}{2!}[S,[S,\mathcal{H}]] + \frac{i^3}{3!}[S,[S,[S,\mathcal{H}]]] + \ldots - \\ \dot{S} - \frac{i}{2!}[S,\dot{S}] - \frac{i^2}{3!}[S,[S,\dot{S}]] - \ldots .\end{array} \tag{3}$$

As a result of this transformation, we find

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta, \tag{4}$$

where the odd operator $\mathcal{O}'$ is now $O(1/m)$. This procedure can be repeated to obtain the required accuracy. As a rule, the classical method of Foldy and Wouthuysen require many successive transformations and a calculation of numerous commutators.

The FW transformation can be performed for particles with any spin.

Relativistic wave equations for particles with any spin can be presented in the Hamilton form. In this case, the Hamilton operator acts on the bispinor wave function $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$:

$$i\hbar\frac{\partial\Psi}{\partial t} = \mathcal{H}\Psi. \tag{5}$$

Explicit form of the Hamilton operators for particles with arbitrary half-integer spin has been obtained in Refs.

A.G. Nikitin, A.V. Galkin. Tensor bispinor equations for doublets. Hadronic J. **26**, 351-366 (2003).

J. Niederle, A.G. Nikitin. Relativistic Coulomb problem for particles with arbitrary half-integer spin. J. Phys. A **39**, 10931-10944 (2006).

The found Hamiltonians are not given in the FW representation and needs additional transformations.

We can introduce the Pauli matrices those components act on the spinors:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 \equiv \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Hamiltonian can be split into operators commuting and non-commuting with the operator β:

$$\mathcal{H} = \beta\mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \beta\mathcal{M} = \mathcal{M}\beta, \quad \beta\mathcal{E} = \mathcal{E}\beta, \quad \beta\mathcal{O} = -\mathcal{O}\beta, \quad (6)$$

where the operators $\mathcal{M}$ and $\mathcal{E}$ are even and the operator $\mathcal{O}$ is odd.

However, general properties of the Hamiltonian depend on the particle spin. The Hamiltonian is hermitian ($\mathcal{H} = \mathcal{H}^\dagger$) for spin-1/2 particles and pseudo-hermitian for spin-0 and spin-1 ones (more precisely, β-pseudo-hermitian, see the reference:)

Ali Mostafazadeh. Is Pseudo-Hermitian Quantum Mechanics an Indefinite-Metric Quantum Theory? Czech. J. Phys. **53**, 1079 (2003).

In the latter case, it possesses the property ($\beta^{-1} = \beta$):

$$\mathcal{H}^\dagger = \beta\mathcal{H}\beta$$

that is equivalent to

$$\mathcal{H}^\ddagger \equiv \beta\mathcal{H}^\dagger\beta = \mathcal{H}.$$

The normalization of wave functions is given by

$$\int \Psi^\dagger\Psi dV = \int (\phi\phi^* + \chi\chi^*)dV = 1$$

for spin-1/2 particles and

$$\int \Psi^\ddagger\Psi dV \equiv \int \Psi^\dagger\beta\Psi dV = \int (\phi\phi^* - \chi\chi^*)dV = 1$$

for spin-0 and spin-1 ones.

We suppose $\mathcal{M}=\mathcal{M}^\dagger,~\mathcal{E}=\mathcal{E}^\dagger,~\mathcal{O}=\mathcal{O}^\dagger$ when $\mathcal{H}=\mathcal{H}^\dagger$ and $\mathcal{M}=\mathcal{M}^\ddagger,~\mathcal{E}=\mathcal{E}^\ddagger,~\mathcal{O}=\mathcal{O}^\ddagger$ when $\mathcal{H}=\mathcal{H}^\ddagger$. These conditions can be satisfied in any case.

The FW transformation ($\Psi_{FW}=U\Psi$) bring the initial Hamiltonian to the block-diagonal (diagonal in two spinors) form and therefore nullifies the odd term. This transformation should be performed with the unitary operator ($U^\dagger=U^{-1}$) for spin-1/2 particles and with the pseudo-unitary operator ($U^\ddagger\equiv\beta U^\dagger\beta=U^{-1}$) for spin-0 and spin-1 ones.

For spin-1/2 particles, the operator $\mathcal{M}$ is equal to the particle mass, m:

$$\mathcal{H}=\beta m+\mathcal{E}+\mathcal{O}. \tag{7}$$

In Eqs. (7)–(11), the system of units $\hbar=c=1$ is used.

When $[\mathcal{E},\mathcal{O}]=0$, the FW transformation is exact:

A. J. Silenko, J. Math. Phys. **44**, 2952 (2003).

This transformation is fulfilled with the operator

$$U = \frac{\epsilon + m + \beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon + m)}}, \quad \epsilon = \sqrt{m^2 + \mathcal{O}^2} \tag{8}$$

and the transformed Hamiltonian takes the form

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}. \tag{9}$$

The same transformation is valid in the more general case when the particle mass m is replaced by $\mathcal{M}$ and the operator $\mathcal{M}$ also commutates with $\mathcal{O}$: $[\mathcal{M}, \mathcal{O}] = 0$. In this case, the precedent formula remains valid but the operator ϵ takes the form

$$\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}.$$

3 FOLDY-WOUTHUYSEN TRANSFORMATION FOR RELATIVISTIC SPIN-1/2 PARTICLES IN EXTERNAL FIELDS

The general FW transformation for relativistic spin-1/2 particles in external fields has been fulfilled in the reference

A. J. Silenko, J. Math. Phys. **44**, 2952 (2003).

The FW Hamiltonian has been obtained as a power series in the external field potentials and their derivatives. As a result of the first stage of transformation, the following Hamiltonian can be found:

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta, \qquad (10)$$

where the odd operator $\mathcal{O}'$ is now comparatively small:

$$\begin{aligned}
\epsilon = \sqrt{m^2+\mathcal{O}^2}, \\
\mathcal{E}' = i\frac{\partial}{\partial t} + \frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}} \\
-\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}, \\
\mathcal{O}' = \frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}} \\
-\frac{\epsilon+m}{\sqrt{2\epsilon(\epsilon+m)}}\left(\mathcal{E}-i\frac{\partial}{\partial t}\right)\frac{\beta\mathcal{O}}{\sqrt{2\epsilon(\epsilon+m)}}.
\end{aligned} \tag{11}$$

The second stage of transformation leads to the approximate equation for the FW Hamiltonian:

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\beta\left\{\mathcal{O}'^2, \frac{1}{\epsilon}\right\}. \tag{12}$$

However, this equation does not define the exact classical limit of the relativistic quantum mechanics of spin-1/2 particles in **strong** external fields.

4 FOLDY-WOUTHYUSEN TRANSFORMATION FOR PARTICLES IN STRONG EXTERNAL FIELDS

It is possible to perform the FW transformation for particles with arbitrary spin in strong external fields by a similar method. This transformation allows us to obtain a meaningful classical limit of the relativistic quantum mechanics. In this case, the FW Hamiltonian can be expanded into a power series in the Planck constant. Since just this constant defines the order of magnitude of quantum corrections, the transition to the semiclassical approximation becomes trivial. Of course, such an expansion can be available only if $pl \gg \hbar$, where p is the momentum of the particle and l is the characteristic size of the nonuniformity region of the external field. This relation is equivalent to $\lambda \ll l$, where λ is the de Broglie wavelength. These relations result from the fact that the Planck constant appears in the final Hamiltonian due to commutators between the operators $\mathcal{M}, \mathcal{E}$, and $\mathcal{O}$.

The expansion of the FW Hamiltonian into a power series in the Planck constant is formally similar to the previously obtained expansion into a power series in the external field potentials and their derivatives. However, only the latter expansion has been fulfilled in the reference

A. J. Silenko, J. Math. Phys. **44**, 2952 (2003).

Therefore, the equations derived in this work do not define the semiclassical limit of the Dirac equation for particles in strong external fields, while these equations exhaustively describe the weak-field approximation. When the power series in the Planck constant is used, zero power terms define the quantum analogue of the classical Hamiltonian. On this level, classical and quantum expressions should be very similar because the classical theory give the right limit of the quantum one. Terms proportional to powers of $\hbar$ may describe quantum corrections. As a rule, interactions described by these terms also exist in the classical theory. However, classical expressions may differ from the corresponding quantum ones because the quantum corrections to the classical theory may appear. All terms in the FW Hamiltonian define the interaction of the particle with the **strong** external fields.

5 METHOD OF FOLDY-WOUTHUYSEN TRANSFORMATION IN STRONG EXTERNAL FIELDS

We generalize the method developed in the references
A. J. Silenko, J. Math. Phys. **44**, 2952 (2003)
and

A. J. Silenko, Zs. Eksp. Teor. Fiz. **123**, 883 (2003) [JETP **96**, 775 (2003)]
in order to take into account a possible non-commutativity of the operators $\mathcal{M}$ and $\mathcal{O}$. The natural generalization of the transformation operator (8) is

$$U = \frac{\beta\epsilon + \beta\mathcal{M} - \mathcal{O}}{\sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}}\,\beta, \quad U^{-1} = \beta\,\frac{\beta\epsilon + \beta\mathcal{M} - \mathcal{O}}{\sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}}. \qquad (13)$$

where $U^\dagger = U^{-1}$ when $\mathcal{H} = \mathcal{H}^\dagger$ and $U^\ddagger = U^{-1}$ when $\mathcal{H} = \mathcal{H}^\ddagger$.

We consider the case when external fields are stationary. In this case, the transformed Hamilton operator is given by

$$\mathcal{H}' = U\mathcal{H}U^{-1}.$$

After the first transformation, the Hamiltonian $\mathcal{H}'$ still contains odd terms proportional to the first and higher powers of the Planck constant. Let us write the operator $\mathcal{H}'$ as

$$\mathcal{H}' = \beta\epsilon + \mathcal{E}' + \mathcal{O}', \quad \beta\mathcal{E}' = \mathcal{E}'\beta, \quad \beta\mathcal{O}' = -\mathcal{O}'\beta, \qquad (14)$$

where $\epsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}$.

In this case, $\mathcal{E}'$ and $\mathcal{O}'$ mean the even and odd parts of the Hamiltonian $\mathcal{H}'$. Additional transformations performed according to reference

A. J. Silenko, J. Math. Phys. **44**, 2952 (2003)

bring the Hamiltonian to the block-diagonal form. The approximate equation for the final FW Hamiltonian is

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E}' + \frac{1}{4}\beta\left\{\mathcal{O}'^2, \frac{1}{\epsilon}\right\}. \tag{15}$$

We have derived the exact formula for the intermediate Hamiltonian $\mathcal{H}'$ which is given by

$$\mathcal{H}' = \beta\epsilon + \mathcal{E} + \frac{1}{2T}([T,[T,(\beta\epsilon+\mathcal{E})]] + \beta\,[\mathcal{O},[\mathcal{O},\mathcal{M}]] \\ -[\mathcal{O},[\mathcal{O},\mathcal{E}]] - [(\epsilon+\mathcal{M}),[(\epsilon+\mathcal{M}),\mathcal{E}]] - [(\epsilon+\mathcal{M}),[\mathcal{M},\mathcal{O}]] \\ -\beta\,\{\mathcal{O},[(\epsilon+\mathcal{M}),\mathcal{E}]\} + \beta\,\{(\epsilon+\mathcal{M}),[\mathcal{O},\mathcal{E}]\})\frac{1}{T}, \tag{16}$$

where $T = \sqrt{(\beta\epsilon + \beta\mathcal{M} - \mathcal{O})^2}$.

The even and odd parts of Hamiltonian (16) are defined by the well-known relations:

$$\mathcal{E}' = \frac{1}{2}\left(\mathcal{H}' + \beta\mathcal{H}'\beta\right) - \beta\epsilon, \quad \mathcal{O}' = \frac{1}{2}\left(\mathcal{H}' - \beta\mathcal{H}'\beta\right).$$

These equations solve the problem of obtaining the semiclassical limit of relativistic quantum mechanics of particles with arbitrary spin in the strong external fields.

Eq. (16) can be significantly simplified in some special cases. When

$[\mathcal{M}, \mathcal{O}] = 0$, it is reduced to

$$\mathcal{H}' = \beta\epsilon + \mathcal{E} + \frac{1}{2T}([T, [T, \mathcal{E}]] \\ - [\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - [(\epsilon + \mathcal{M}), [(\epsilon + \mathcal{M}), \mathcal{E}]] \\ -\beta \{\mathcal{O}, [(\epsilon + \mathcal{M}), \mathcal{E}]\} + \beta \{(\epsilon + \mathcal{M}), [\mathcal{O}, \mathcal{E}]\})\frac{1}{T}. \tag{17}$$

In this case, $[\epsilon, \mathcal{M}] = [\epsilon, \mathcal{O}] = 0$. The operator T is even and is given by

$$T = \sqrt{2\epsilon(\epsilon + \mathcal{M})}.$$

6 EXAMPLE 1: DIRAC PARTICLES IN STRONG ELECTROMAGNETIC FIELD

As an example, we consider the interaction of Dirac particles with a strong electromagnetic field. The initial Dirac-Pauli Hamiltonian for a particle possessing an anomalous magnetic moment (AMM) has the form:

W. Pauli, Rev. Mod. Phys. **13**, 203 (1941)

$$\mathcal{H}_{DP} = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta mc^2 + e\Phi + \mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H} + i\boldsymbol{\gamma}\cdot\boldsymbol{E}),$$
$$\boldsymbol{\pi} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}, \quad \mu' = \frac{g-2}{2}\cdot\frac{e\hbar}{2mc}, \tag{18}$$

where μ' is the AMM, $\Phi, \boldsymbol{A}$ and $\boldsymbol{E}, \boldsymbol{H}$ are the potentials and the strengths of the electromagnetic field.

Terms describing an electric dipole moment (EDM) have been added in the reference

A. J. Silenko, Izvestiya vuz. Fizika, No. 8, 9 (2005) [Russian Physics Journal, **48**, 788 (2005)].

The resulting Hamiltonian is given by

$$\mathcal{H} = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + \beta mc^2 + e\Phi + \mu'(-\boldsymbol{\Pi}\cdot\boldsymbol{H} + i\boldsymbol{\gamma}\cdot\boldsymbol{E})$$
$$-d(\boldsymbol{\Pi}\cdot\boldsymbol{E} + i\boldsymbol{\gamma}\cdot\boldsymbol{H}), \quad d = \frac{\eta}{2}\cdot\frac{e\hbar}{2mc}, \tag{19}$$

where η factor is an analogue of g factor.

It is important that μ' and d are proportional to $\hbar$.

In this case

$$\mathcal{M} = mc^2, \quad \mathcal{E} = e\Phi - \mu'\mathbf{\Pi}\cdot\boldsymbol{H} - d\mathbf{\Pi}\cdot\boldsymbol{E},$$
$$\mathcal{O} = c\boldsymbol{\alpha}\cdot\boldsymbol{\pi} + i\mu'\boldsymbol{\gamma}\cdot\boldsymbol{E} - id\boldsymbol{\gamma}\cdot\boldsymbol{H}.$$

When calculating the FW Hamiltonian, we retain only terms of zero and first powers in the Planck constant because just such terms define the quantum equations of motion of particles and their spins.

The obtained Hamiltonian is given by

$$
\begin{aligned}
\mathcal{H}_{FW} = \beta\epsilon' + e\Phi - \mu'\mathbf{\Pi}\cdot\boldsymbol{H} - \frac{\mu_0}{2}\left\{\frac{mc^2}{\epsilon'}, \mathbf{\Pi}\cdot\boldsymbol{H}\right\} \\
+\frac{\mu' c}{4}\left\{\frac{1}{\epsilon'}, [\mathbf{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E}) - \mathbf{\Sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi})]\right\} \\
+\frac{\mu_0 mc^3}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}[\mathbf{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{E}) - \mathbf{\Sigma}\cdot(\boldsymbol{E}\times\boldsymbol{\pi})]\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
+\frac{\mu' c^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\mathbf{\Pi}\cdot\boldsymbol{\pi}), (\boldsymbol{H}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{H})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
-d\mathbf{\Pi}\cdot\boldsymbol{E} \\
+\frac{dc^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\mathbf{\Pi}\cdot\boldsymbol{\pi}), (\boldsymbol{E}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{E})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
-\frac{dc}{4}\left\{\frac{1}{\epsilon'}, [\mathbf{\Sigma}\cdot(\boldsymbol{\pi}\times\boldsymbol{H}) - \mathbf{\Sigma}\cdot(\boldsymbol{H}\times\boldsymbol{\pi})]\right\},
\end{aligned}
\tag{20}
$$

where

$$\epsilon' = \sqrt{m^2c^4 + c^2\boldsymbol{\pi}^2} \tag{21}$$

and $\mu_0 = \dfrac{e\hbar}{2mc}$ is the Dirac magnetic moment.

The operator equation of particle motion defines the quantum evolution of the kinetic momentum operator, $\boldsymbol{\pi}$:

$$\frac{d\boldsymbol{\pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \boldsymbol{\pi}] - \frac{e}{c} \cdot \frac{\partial \boldsymbol{A}}{\partial t}.$$

The equation of spin motion describes the evolution of the polarization operator, $\mathbf{\Pi}$:

$$\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{FW}, \mathbf{\Pi}].$$

The equation of particle motion can be divided into two parts. The first of them does not contain the Planck constant and describes the quantum equivalent of the Lorentz force. The second part is of order of $\hbar$. This part is proportional either to field gradients or to products of field strengths (H^2, E^2 and EH).

Because the operator $\boldsymbol{\pi}$ does not contain the Dirac spin matrices, the commutation of this operator with the Hamiltonian brings the additional factor $\hbar$. To deduce the equation of particle motion to within first-order terms in the Planck constant, one therefore has to retain terms of the same order in the Hamiltonian. The equation of spin-1/2 particle motion in the strong electromagnetic form has the form

$$
\begin{aligned}
\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \beta\frac{e}{4c}\left\{\frac{1}{\epsilon'}, ([\boldsymbol{\pi}\times\boldsymbol{H}] - [\boldsymbol{H}\times\boldsymbol{\pi}])\right\} \\
+\mu'\nabla(\boldsymbol{\Pi}\cdot\boldsymbol{H}) + \frac{\mu_0}{2}\left\{\frac{mc^2}{\epsilon'}, \nabla(\boldsymbol{\Pi}\cdot\boldsymbol{H})\right\} \\
-\frac{\mu'c}{4}\left\{\frac{1}{\epsilon'}, [\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]) - \nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])]\right\} \\
-\frac{\mu_0 mc^3}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}[\nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]) - \nabla(\boldsymbol{\Sigma}\cdot[\boldsymbol{E}\times\boldsymbol{\pi}])]\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}} \\
-\frac{\mu'c^2}{2\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Pi}\cdot\boldsymbol{\pi}), [\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi}) + \nabla(\boldsymbol{\pi}\cdot\boldsymbol{H})]\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}.
\end{aligned}
\tag{22}
$$

In this equation, terms proportional to products of field strengths are not shown because they are usually small in comparison with the terms retained in Eq. (22). The part of the equation proportional to $\hbar$ defines the relativistic expression for the Stern-Gerlach force. The terms proportional to d are omitted.

The equation of spin motion is given by

$$\begin{aligned}\frac{d\boldsymbol{\Pi}}{dt} &= 2\mu'\boldsymbol{\Sigma}\times\boldsymbol{H}+\mu_0\left\{\frac{mc^2}{\epsilon'},\boldsymbol{\Sigma}\times\boldsymbol{H}\right\}\\ &-\frac{\mu'c}{2}\left\{\frac{1}{\epsilon'},[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{E})-\boldsymbol{\Pi}\times(\boldsymbol{E}\times\boldsymbol{\pi})]\right\}\\ &-\frac{\mu_0 mc^3}{\sqrt{\epsilon'(\epsilon'+mc^2)}}[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{E})-\boldsymbol{\Pi}\times(\boldsymbol{E}\times\boldsymbol{\pi})]\frac{1}{\sqrt{\epsilon'(\epsilon'+mc^2)}}\\ &-\frac{\mu'c^2}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Sigma}\times\boldsymbol{\pi}),(\boldsymbol{H}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{H})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}\\ &+2d\boldsymbol{\Sigma}\times\boldsymbol{E}\\ &-\frac{dc^2}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}\{(\boldsymbol{\Sigma}\times\boldsymbol{\pi}),(\boldsymbol{E}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{E})\}\frac{1}{\sqrt{2\epsilon'(\epsilon'+mc^2)}}\\ &+\frac{dc}{2}\left\{\frac{1}{\epsilon'},[\boldsymbol{\Pi}\times(\boldsymbol{\pi}\times\boldsymbol{H})-\boldsymbol{\Pi}\times(\boldsymbol{H}\times\boldsymbol{\pi})]\right\}.\end{aligned}\tag{23}$$

7 EXAMPLE 2: SPINLESS PARTICLES IN STRONG ELECTROMAGNETIC FIELD

The initial Klein-Gordon equation for particles in the strong electromagnetic field has been transformed to the Hamilton form in the reference

H. Feshbach and F. Villars. Elementary relativistic wave mechanics of spin 0 and spin 1/2 particles. Rev. Mod. Phys. 1958. V. 30. P. 24–45.

The Hamiltonian acts on the two-component wave function which is the analogue of the spinor for spin-0 particles. The explicit form of the Hamiltonian is

$$\mathcal{H} = \rho_3 mc^2 + (\rho_3 + i\rho_2)\frac{\boldsymbol{\pi}^2}{2m} + e\Phi. \tag{24}$$

Thus,

$$\mathcal{M} = mc^2 + \frac{\boldsymbol{\pi}^2}{2m}, \quad \mathcal{E} = e\Phi, \quad \mathcal{O} = i\rho_2 \frac{\boldsymbol{\pi}^2}{2m}, \quad [\mathcal{M}, \mathcal{O}] = 0. \quad (25)$$

In this case

$$\epsilon = \sqrt{m^2c^4 + c^2\boldsymbol{\pi}^2}, \quad T = \sqrt{\frac{\epsilon}{mc^2}}\left(\epsilon + mc^2\right). \quad (26)$$

The Hamiltonian transformed to the FW representation is given by

$$\mathcal{H}_{FW} = \beta\epsilon + \mathcal{E} = \beta\sqrt{m^2c^4 + c^2\boldsymbol{\pi}^2} + e\Phi. \quad (27)$$

There are not any terms of order of $\hbar$ in this Hamiltonian, while it contains terms of second and higher orders in the Planck constant.

The operator equation of particle motion takes the form

$$\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \beta\frac{e}{4c}\left\{\frac{1}{\sqrt{m^2 + \boldsymbol{\pi}^2}}, ([\boldsymbol{\pi} \times \boldsymbol{H}] - [\boldsymbol{H} \times \boldsymbol{\pi}])\right\}. \quad (28)$$

This equation coincides with the spin-independent part of the corresponding equation for spin-1/2 particles.

8 SEMICLASSICAL LIMIT OF RELATIVISTIC QUANTUM MECHANICS OF PARTICLES IN STRONG EXTERNAL FIELDS

To obtain the semiclassical limit of the relativistic quantum mechanics, one needs to average the operators in the considered equations. This procedure can be successfully performed when commutators of dynamical variables are negligible as compared with the product of average values of these dynamical variables. If the momentum and position operators are chosen to be the dynamical variables, this condition takes the form

$$|<p_i>|\cdot|<x_i>|\gg|<[p_i,x_i]>|=\hbar,\quad i=1,2,3.$$

The angular brackets which designate averaging in time will be hereinafter omitted.

Because of the above mentioned properties of the FW representation, averaging the operators in the quantum equations of motion of particles and their spins is trivial. It consists in the substitution of classical quantities for the corresponding operators. Of course, the obtained equations are just semiclassical and can differ from the corresponding classical equations.

As a result of such a substitution, the semiclassical equations of motion of particles and their spins take the form

$$\frac{d\boldsymbol{\pi}}{dt} = e\boldsymbol{E} + \frac{e}{c\epsilon'}(\boldsymbol{\pi}\times\boldsymbol{H}) + \mu'\nabla(\boldsymbol{P}\cdot\boldsymbol{H}) + \frac{\mu_0}{mc^2\epsilon'}\nabla(\boldsymbol{P}\cdot\boldsymbol{H})$$
$$-\frac{\mu' c}{\epsilon'}\nabla(\boldsymbol{P}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}]) - \frac{\mu_0 mc^3}{\epsilon'(\epsilon'+mc^2)}\nabla(\boldsymbol{P}\cdot[\boldsymbol{\pi}\times\boldsymbol{E}])$$
$$-\frac{\mu' c^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\cdot\boldsymbol{\pi})\nabla(\boldsymbol{H}\cdot\boldsymbol{\pi}), \qquad \boldsymbol{P} = \frac{\boldsymbol{S}}{S}, \tag{29}$$

$$\frac{d\boldsymbol{P}}{dt} = 2\mu'\boldsymbol{P}\times\boldsymbol{H} + \frac{2\mu_0 mc^2}{\epsilon'}(\boldsymbol{P}\times\boldsymbol{H}) - \frac{2\mu' c}{\epsilon'}(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{E}])$$
$$-\frac{2\mu_0 mc^3}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{E}]) - \frac{2\mu' c^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{H})$$
$$+2d\boldsymbol{P}\times\boldsymbol{E} - \frac{2dc^2}{\epsilon'(\epsilon'+mc^2)}(\boldsymbol{P}\times\boldsymbol{\pi})(\boldsymbol{\pi}\cdot\boldsymbol{E}) + \frac{2dc}{\epsilon'}(\boldsymbol{P}\times[\boldsymbol{\pi}\times\boldsymbol{H}])\,. \tag{30}$$

In these equations, $\boldsymbol{P}$ is the polarization vector and $\boldsymbol{S}$ is the spin vector (i.e., the average spin).

The spin-independent part of the equation of spin motion coincides with the well-known Bargmann-Michel-Telegdi (BMT) equation. It is natural because the BMT equation has been derived without the assumption that the external fields are weak. Two first terms in the equation of particle motion are the same as in the classical expression for the Lorentz force. Because the relativistic formula for the Stern-Gerlach force can be obtained from the BMT equation, the semiclassical and classical relations for this force also coincide.

Newertheless, high-order corrections to the quantum equations of motion of particles and their spins should bring a difference between quantum and classical approaches.

9 CONCLUSIONS

• The FW transformation for relativistic particles of arbitrary spin in strong external fields has been fulfilled. The exact description of spin dynamics in strong external fields has been performed.

• The transition to the semiclassical approximation has been made.

• The quantum and semiclassical equations of motion of Dirac particles and their spins and the equation of motion of spinless particles in strong electromagnetic fields have been derived.

• The best agreement between the quantum and classical approaches has been established.