Time-dependent exactly solvable models and quantum computing

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- Construction of time-dependent exactly solvable models. A class of periodic time-dependent Hamiltonians with cyclic solutions is constructed in a closed analytic form.
- The method is based on transformation of soluble time-independent Eqs. into time-dependent ones by employing a set of special time-dependent transformation operators.
- A time-dependent periodic Hamiltonian admitting exact solutions is applied to construct a set of universal gates for quantum computer.
- The time evolution matrices are obtained in an explicit form and used to construct logic gates for computation. A way of obtaining entanglement operator is discussed, too.

Construction of a time-dependent Hamiltonian

Suppose that the state $|\Psi(r,t)\rangle$ of a dynamical system evolves according to the matrix Schrödinger equation

$$i\frac{d|\Psi(r,t)\rangle}{dt} = H(r,t)|\Psi(r,t)\rangle, \qquad \tilde{H} = \hat{p}_r^2 + V(r) \tag{1}$$

with T - periodic time-dependent Hamiltonian, ***H(t)=H(t+T),*** $\hbar^2/2m = 1$. The potential matrix $V(r,t) = \{V_{ij}(r,t)\}$ is Hermitian and p_r is the momentum operator.

Our goal is to give the procedure for obtaining a wide class of time-dependent Hamiltonians H(t) for which exact solutions of (1) can be found. To this end, we use the time-independent Hamiltonian

$$\tilde{H} = \hat{p}_r^2 + V(r) \tag{2}$$

with a real symmetric potential matrix $V(r)$ and a unitary time-dependent transformation **S(t)**

$$|\Psi(r,t)\rangle = S(t)|\Phi(r,t)\rangle \tag{3}$$

by means of which the known time-independent Hamiltonian (2) is changed to the time-dependent one

$$H(t) = S(t)\tilde{H}S^+(t) + i\dot{S}(t)S^+(t) \tag{4}$$

Here $|\Phi(\mathbf{r},\mathbf{t})\rangle$ satisfies the Eq. of motion $\mathbf{i}\frac{\mathbf{d}|\Phi(\mathbf{r},\mathbf{t})\rangle}{\mathbf{dt}}=\widetilde{\mathbf{H}}(\mathbf{r})|\Phi(\mathbf{r},\mathbf{t})\rangle$ with the time-independent Hamiltonian $\widetilde{\mathbf{H}}(\mathbf{r})$ and it is of the form

$$|\Phi(\mathbf{r},\mathbf{t})\rangle=\exp\left(-\mathbf{i}\widetilde{\mathbf{H}}(\mathbf{r})\mathbf{t}\right)|\Phi(\mathbf{r},0)\rangle \tag{5}$$

Clearly, the solutions $|\Psi(\mathbf{r},\mathbf{t})\rangle$ and $|\Phi(\mathbf{r},\mathbf{t})\rangle$ can be properly defined by solutions of the time-independent problem

$$\widetilde{\mathbf{H}}|\Phi(\widetilde{\varepsilon})\rangle=\widetilde{\varepsilon}|\Phi(\widetilde{\varepsilon})\rangle \tag{6}$$

The technique of canonical transformation from a time-independent Hamiltonian to a time-dependent one was used: for rotating systems by
S.J. Wang, Phys. Rev. ***A 42*** *(1990) 5107; ibid p. 5103,*
Shi-Min Cui, Phys.Rev. A 45 (1992) 5255,
Ali Mostafazadeh: rotating systems and oscillators.

Note, if the system of Schrödinger equations (6) with some known time-independent Hamiltonian $\widetilde{\mathbf{H}}$ ***is exactly soluble, the system of equations (1) with the time-dependent Hamiltonian*** $\mathbf{H}(\mathbf{t})=\mathbf{S}(\mathbf{t})\widetilde{\mathbf{H}}\mathbf{S}^{+}(\mathbf{t})+\mathbf{i}\dot{\mathbf{S}}(\mathbf{t})\mathbf{S}^{+}(\mathbf{t})$ ***(4) admits exact solutions, too. The result depends on transformation operators*** $\mathbf{S}(\mathbf{t})$ ***and the choice of initial states.*** Now consider reconstruction of the $\mathbf{2}\times\mathbf{2}$ periodic time-dependent Hamiltonian. taken in the form (4).

We start with the time-independent Hamiltonian $\widetilde{\mathbf{H}}(\mathbf{r})$ with the $\mathbf{2\times 2}$ real symmetric potential matrix $\mathbf{V_{12}(r) = V_{21}(r)}$. By means of a unitary time-dependent transformation taken in the form

$$\mathbf{S(t) = \exp(-is\cdot h(t)) = \exp\left(-i\sum_{i=1}^{3} s_i h_i(t)\right)} \tag{7}$$

the time-independent Hamiltonian $\widetilde{\mathbf{H}} = \hat{\mathbf{p}}_{\mathbf{r}}^{\mathbf{2}} + \mathbf{V(r)}$ with regard to Schr.Eqs. (1) and $\mathbf{H(t) = S(t)\widetilde{H}S^{+}(t) + i\dot{S}(t)S^{+}(t)}$ turns to the time-dependent Hamiltonian

$$\mathrm{H}(\mathrm{r,t}) = \mathrm{p}_{\mathrm{r}}^{2} + \exp(-\mathbf{is}\cdot \mathrm{h}(\mathrm{t}))\mathrm{V}(\mathrm{r})\exp(\mathbf{is}\cdot \mathbf{h}(\mathrm{t})) + (\mathbf{s}\cdot \dot{\mathbf{h}}(\mathrm{t})) \tag{8}$$

The solutions of (1) with Hamiltonian (8), according to $\mathbf{|\Psi(r,t)\rangle = S(t)|\Phi(r,t)\rangle}$ and $\mathbf{|\Phi(r,t)\rangle = \exp(-i\widetilde{H}(r)t)|\Phi(r,0)\rangle}$, are represented as

$$\mathbf{|\Psi(r,t)\rangle = \exp(-is\cdot h(t))\exp(-i\widetilde{H}(r)t)|\Phi(r,t=0)\rangle} \tag{9}$$

Here $\mathbf{s} = (1/2)\boldsymbol{\sigma}$ is the spin operator, $\boldsymbol{\sigma} = (\hat{\boldsymbol{\sigma}}_1, \hat{\boldsymbol{\sigma}}_2, \hat{\boldsymbol{\sigma}}_3)$ and $\hat{\boldsymbol{\sigma}}_{\mathbf{i}}$ are the Pauli matrices and a dot means a time-derivative.

It is convenient to present the $\mathbf{2\times 2}$ intrinsic time-independent Hamiltonian (2) by the sum of diagonal and zero trace matrices:

$$\begin{aligned}\widetilde{\mathbf{H}}(\mathbf{r}) &= \mathbf{p}_\mathbf{r}^2+\begin{pmatrix}\mathbf{V}_{11}(\mathbf{r}) & \mathbf{V}_{12}(\mathbf{r})\\ \mathbf{V}_{21}(\mathbf{r}) & \mathbf{V}_{22}(\mathbf{r})\end{pmatrix}=\\ &=\left(\mathbf{p}_\mathbf{r}^2+\frac{\mathbf{V}_{11}(\mathbf{r})+\mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}}\right)\hat{\mathbf{I}}+\begin{pmatrix}\dfrac{\mathbf{V}_{11}(\mathbf{r})-\mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}} & \mathbf{V}_{12}(\mathbf{r})\\ \mathbf{V}_{21}(\mathbf{r}) & -\dfrac{\mathbf{V}_{11}(\mathbf{r})-\mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}}\end{pmatrix}\end{aligned} \tag{10}$$

$$\widetilde{\mathbf{H}}(\mathbf{r})=\left(\mathbf{p}_\mathbf{r}^2+\mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}}+\mathbf{2}(\mathbf{s}\cdot\mathbf{B}(\mathbf{r})) \tag{11}$$

with the evident notations: $\mathbf{q}(\mathbf{r})=\dfrac{\mathbf{V}_{11}(\mathbf{r})+\mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}}$, $\mathbf{B}_1(\mathbf{r})=\mathbf{V}_{12}(\mathbf{r})$, $\mathbf{B}_2(\mathbf{r})=\mathbf{0}$, $\mathbf{B}_3(\mathbf{r})=\dfrac{\mathbf{V}_{11}(\mathbf{r})-\mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}}$ and $\hat{I}$ is the identity matrix. It is evident that the Hamiltonian for the two coupled systems of Eqs. corresponds to the three- or two-dimensional problem with coordinates $\mathbf{B}_\mathbf{i}$ dependent on the extra parameter r. Then the time-dependent Hamiltonian (8) can be represented as

$$\begin{aligned}\mathbf{H}(\mathbf{r},\mathbf{t}) &= \left(\mathbf{p}_\mathbf{r}^2+\mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}}+\mathbf{2}\exp(-\mathbf{is}\cdot\mathbf{h}(\mathbf{t}))(\mathbf{s}\cdot\mathbf{B}(\mathbf{r}))\exp(\mathbf{is}\cdot\mathbf{h}(\mathbf{t}))+(\mathbf{s}\cdot\dot{\mathbf{h}}(\mathbf{t}))=\\ &=\left(\mathbf{p}_\mathbf{r}^2+\mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}}+\mathbf{2}(\mathbf{s}\cdot\mathbf{B}(\mathbf{r},\mathbf{t}))\end{aligned} \tag{12}$$

Obviously, the transformation $\mathbf{S}(\mathbf{t})=\exp(-\mathbf{is}\cdot\mathbf{h}(\mathbf{t}))$ does not change the first term of (10) or (11) and transforms the second term. The Hamiltonians in the forms (12) and (8) can be used for describing the motion of a spin 1/2 -particle in the space-nonuniform and time-dependent magnetic field or can be applied for investigating multi-level atoms and nuclei.

In terms of the evolution operator $\mathbf{U(t)=U(t,0)}$ the solution $|\boldsymbol{\Psi}(\mathbf{r},\mathbf{t})\rangle$ is

$$|\boldsymbol{\Psi}(\mathbf{r},\mathbf{t})\rangle=\mathbf{U(t)}|\boldsymbol{\Psi}(\mathbf{r},\mathbf{0})\rangle, \qquad \mathbf{U(0)=1} \tag{13}$$

It is easy to find from $|\boldsymbol{\Psi}(\mathbf{r},\mathbf{t})\rangle=\mathbf{exp(-is\cdot h(t))exp(-i\tilde{H}(r)t)}|\boldsymbol{\Phi}(\mathbf{r},\mathbf{t=0})\rangle$ (9) and (13) a very important relationship between the operators $\mathbf{U(t)}$ and $\mathbf{S(t)}$ In the case when $|\boldsymbol{\Psi}(\mathbf{r},\mathbf{0})\rangle=|\boldsymbol{\Phi}(\mathbf{r},\mathbf{0})\rangle$

$$\mathbf{U(t)=S(t)exp\left(-i\tilde{H}t\right)=exp(is\cdot h(t))exp\left(-i\tilde{H}t\right)} \tag{14}$$

The evolution operator in one period is written as $\mathbf{U(T)=exp(is\cdot h(T))exp\left(-i\tilde{H}T\right)}$ (15)

Now let us consider cyclic solutions that after one period $T(T=2\pi/\omega)$ are recovered up to the phase, i.e., initial states $|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{0})\rangle$ are eigenvectors of $\mathbf{U(T)}$

$$|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{T})\rangle=\mathbf{U(T)}|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{0})\rangle=\mathbf{exp}(-\mathbf{i}\boldsymbol{\beta}_{\mathbf{v}})|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{0})\rangle \tag{16}$$

where $\mathbf{exp}(-\mathbf{i}\boldsymbol{\beta}_{\mathbf{v}})$ are eigenvalues of $\mathbf{U(T)}$ and $\boldsymbol{\beta}_{\mathbf{v}}$ is the total phase. ***Let us demand that initial states are eigenvectors of the time-independent Hamiltonian*** $\tilde{\mathbf{H}}$***,*** $|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{0})\rangle=|\boldsymbol{\Phi}_{\mathbf{v}}(\mathbf{r})\rangle$***. It is possible if*** $\mathbf{U(T)}$ ***and*** $\tilde{\mathbf{H}}$ ***commute*** $\left[\mathbf{U(T)},\tilde{\mathbf{H}}\right]=\mathbf{0}$. ***Then, we immediately obtain that*** $\mathbf{S(T)}$ ***and*** $\tilde{\mathbf{H}}$ ***commute*** $\left[\mathbf{S(T)},\tilde{\mathbf{H}}\right]=\mathbf{0}$ (17). It is one of the conditions on the choice of transformations $\mathbf{S(t)}$. Other properties of $\mathbf{S(t)}$ are evident: they have to be unitary and possess the same matrix dimension as $\tilde{H}$. With allowance for (9) the recurrent solutions at any time are written as

$$|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{t})\rangle=\mathbf{exp(-is\cdot h(t))exp\left(-i\tilde{E}_{v}t\right)}|\boldsymbol{\Phi}_{\mathbf{v}}(\mathbf{r})\rangle \tag{18}$$

It is evident now that in order to determine the cyclic solutions, we need time-independent solutions. Thus, whenever $\mathbf{\widetilde{H}(r)}$ is an exactly soluble time-independent Hamiltonian, the properly generated time-dependent system of Eqs. has cyclic exact solutions. The evolution of an arbitrary initial state $\left|\boldsymbol{\Psi}(\mathbf{r},\mathbf{0})\right\rangle = \sum_{\mathbf{v}} \boldsymbol{\alpha}_{\mathbf{v}} \left|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{t})\right\rangle$ can be represented as a superposition of a basis set of recurrent linearly independent vector solutions,

$$\left|\boldsymbol{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{t})\right\rangle = \mathbf{exp}(-\mathbf{is}\cdot\mathbf{h(t)})\mathbf{exp}\left(-\mathbf{i}\widetilde{\mathbf{E}}_{\mathbf{v}}\mathbf{t}\right)\left|\boldsymbol{\Phi}_{\mathbf{v}}(\mathbf{r})\right\rangle$$

Exactly solvable 2×2 time-dependent potential matrices

Let us consider some particular examples of constructing a time-dependent Hamiltonian with corresponding solutions by using **SU(2)** transformation *(7)* $\mathbf{S(t)=\exp(-is\cdot h(t))}$ *in which the components of* **h** *are linear functions of time,* $\mathbf{h_i(t)=\omega_i t}$

$\mathbf{S_1(t)=\exp(-i\hat{\sigma}_1\omega_1 t/2)}$, $\mathbf{S_2(t)=\exp(-i\hat{\sigma}_2\omega_2 t/2)}$, $\mathbf{S_3(t)=\exp(-\hat{\sigma}_3\omega t/2)}$

1 case. ***Let*** S(t) ***be chosen as an operator rotation around z-axis***

$$\mathbf{S_3(t)=\exp(-\hat{\sigma}_3\omega t/2)=\begin{pmatrix}\exp(-i\omega t/2) & 0\\ 0 & \exp(i\omega t/2)\end{pmatrix}} \qquad (19)$$

where ω is some constant angular velocity. Then, in accordance with $\mathbf{H(t)=p_r^2+\exp(-ish(t))V(r)\exp(is\cdot h(t))+(s\cdot\dot{h}(t))}$ the time-independent Hamiltonian (10) is transformed into the time-dependent one as

$$\mathbf{H(r,t)=(p_r^2+q(r))\hat{I}+\begin{pmatrix}\dfrac{V_{11}(r)-V_{22}(r)+\omega}{2} & V_{12}(r)\exp(-i\omega t)\\ V_{21}(r)\exp(i\omega t) & -\dfrac{V_{11}(r)-V_{22}(r)+\omega}{2}\end{pmatrix}} \qquad (20)$$

It is evident that $\mathbf{H(t)}$ is $\mathbf{T}$-periodic, $\mathbf{H(r,t=0)=H(r,t=T)}$, $\mathrm{T}=\mathbf{2\pi/\omega}$. Hamiltonian $\mathbf{\widetilde{H}(r)}$ is connected with $\mathbf{H(r,t=0)}$ by $\mathbf{H(r,0)=\widetilde{H}(r)+\dfrac{\omega}{2}\hat{\sigma}_3}$.

By using $\left|\mathbf{\Psi}_{\mathbf{v}}\right\rangle = \mathbf{S(t)exp}\left(-\mathbf{i}\widetilde{\mathbf{E}}_{\mathbf{v}}\mathbf{t}\right)\left|\mathbf{\Phi}_{\mathbf{v}}(\mathbf{r})\right\rangle$ the cyclic solutions can be written as

$$\left|\mathbf{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{t})\right\rangle = \begin{pmatrix} \mathbf{exp}\left(-\mathbf{i}\left(\widetilde{\boldsymbol{\varepsilon}}_1^{\mathbf{v}} + \boldsymbol{\omega}/\mathbf{2}\right)\mathbf{t}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{exp}\left(-\mathbf{i}\left(\widetilde{\boldsymbol{\varepsilon}}_1^{\mathbf{v}} - \boldsymbol{\omega}/\mathbf{2}\right)\mathbf{t}\right) \end{pmatrix} \left|\mathbf{\Phi}_{\mathbf{v}}(\mathbf{r})\right\rangle \qquad (21)$$

and after one period

$$\left|\mathbf{\Psi}_{\mathbf{v}}(\mathbf{r},(\mathbf{T}+\mathbf{t}))\right\rangle = \begin{pmatrix} \mathbf{exp}\left(-\mathbf{i}\widetilde{\boldsymbol{\varepsilon}}_1^{\mathbf{v}}\mathbf{T} - \mathbf{i}\boldsymbol{\pi}\right) & \mathbf{0} \\ \mathbf{0} & \mathbf{exp}\left(-\mathbf{i}\widetilde{\boldsymbol{\varepsilon}}_2^{\mathbf{v}}\mathbf{T} + \mathbf{i}\boldsymbol{\pi}\right) \end{pmatrix} \left|\mathbf{\Psi}_{\mathbf{v}}(\mathbf{r},\mathbf{t})\right\rangle$$

where $\widetilde{\boldsymbol{\varepsilon}}_{\boldsymbol{\alpha}}^{\mathbf{v}} = \left(\mathbf{E}_{\mathbf{v}} - \boldsymbol{\Delta}_{\boldsymbol{\alpha}}\right)$ and Δ_{α} can be the energy of the inner structure (threshold). If $\Delta_{\alpha} = 0$, then the eigenstates $\left|\Phi_{v}(r)\right\rangle$ are twofold degenerated.

If there is no dependence on the space variable, our problem is simplified.

In this case the time-independent Hamiltonian $\widetilde{\mathbf{H}}$ is

$$\widetilde{\mathbf{H}} = \boldsymbol{\sigma}\cdot\widetilde{\mathbf{B}} = \widetilde{\mathbf{\Omega}}\begin{pmatrix} \mathbf{cos}\widetilde{\boldsymbol{\theta}} & \mathbf{sin}\widetilde{\boldsymbol{\theta}} \\ \mathbf{sin}\widetilde{\boldsymbol{\theta}} & -\mathbf{cos}\widetilde{\boldsymbol{\theta}} \end{pmatrix} \qquad (22)$$

with the renormalized uniform magnetic field $\widetilde{\mathbf{B}} = \widetilde{\mathbf{\Omega}}\left(\mathbf{sin}\widetilde{\boldsymbol{\theta}},\mathbf{0},\mathbf{cos}\widetilde{\boldsymbol{\theta}}\right)$

The two eigenvector solutions and eigenvalues of (22) are

$$\left|\mathbf{\Phi}_{\mathbf{1}}\right\rangle = \begin{pmatrix} \mathbf{cos}\left(\widetilde{\boldsymbol{\theta}}/\mathbf{2}\right) \\ \mathbf{sin}\left(\widetilde{\boldsymbol{\theta}}/\mathbf{2}\right) \end{pmatrix} \qquad \left|\mathbf{\Phi}_{\mathbf{2}}\right\rangle = \begin{pmatrix} -\mathbf{sin}\left(\widetilde{\boldsymbol{\theta}}/\mathbf{2}\right) \\ \mathbf{cos}\left(\widetilde{\boldsymbol{\theta}}/\mathbf{2}\right) \end{pmatrix} \qquad \widetilde{\boldsymbol{\varepsilon}}_{\mathbf{1}} = +\widetilde{\mathbf{\Omega}}, \qquad \widetilde{\boldsymbol{\varepsilon}}_{\mathbf{2}} = -\widetilde{\mathbf{\Omega}}. \qquad (23)$$

The time-dependent Hamiltonian for the spin $\mathbf{s = 1/2}$ particle in a rotating but space-homogeneous magnetic field $\mathbf{B(t)}$ becomes

$$\mathbf{H(t) = \sigma \cdot B(t)} = \tilde{\boldsymbol{\Omega}} \begin{pmatrix} \mathbf{\cos\tilde{\theta} + \omega/2\tilde{\Omega}} & \mathbf{\sin\tilde{\theta}\exp(-i\omega}\mathrm{t}) \\ \mathbf{\sin\tilde{\theta}\exp(}\mathrm{i}\boldsymbol{\omega}\mathrm{t}) & \mathbf{-\cos\tilde{\theta} - \omega/2\tilde{\Omega}} \end{pmatrix} \tag{24}$$

$$\mathbf{B(t)} = \tilde{\boldsymbol{\Omega}}\left(\mathbf{\sin\tilde{\theta}\cos(\omega}\mathrm{t}\mathbf{)}, \mathbf{\sin\tilde{\theta}\sin(\omega}\mathrm{t}\mathbf{)\cos(\tilde{\theta}} + \frac{\boldsymbol{\omega}}{\mathbf{2\tilde{\Omega}}}) \right)$$

There are two cyclic solutions which in terms of the above time-independent solutions according to (21) evolve as

$$\left|\boldsymbol{\Psi}_1(\mathbf{t})\right\rangle = \begin{pmatrix} \mathbf{\exp[-i\omega}\mathrm{t}\,/\,\mathbf{2 - i\tilde{\Omega}t]\cos(\tilde{\theta}/2)} \\ \mathbf{\exp[}\mathrm{i}\mathbf{\omega t/2 - i\tilde{\Omega}t]\sin(\tilde{\theta}/2)} \end{pmatrix} \qquad \left|\boldsymbol{\Psi}_2(\mathbf{t})\right\rangle = \begin{pmatrix} \mathbf{\exp[-i\omega}\mathrm{t}\,/\,2 + \mathbf{i\tilde{\Omega}t]\sin(\tilde{\theta}/2)} \\ \mathbf{\exp[}\mathrm{i}\mathbf{\omega t/2 + i\tilde{\Omega}t]\cos(\tilde{\theta}/2)} \end{pmatrix} \tag{25}$$

It is a well known problem of a spin-1/2 particle evolving in a space-homogeneous magnetic field *(see e.g. M. Stone, Phys. Rev. D,* ***33*** *(1987) 1191. S.J. Wang, Phys. Rev. A42 (1990) 5107; ibid p. 5103)* obtained here as a particular case of the problem of a spin-1/2 particle evolving in a periodic time-dependent but space-inhomogeneous magnetic field.

2 case. ***If*** $S(t)$ ***is chosen as an operator of rotation around y-axis***

$$S_2(t) = \exp(-i\hat{\sigma}_2\omega_2 t/2) = \begin{pmatrix} \cos(\omega_2 t/2) & \sin(\omega_2 t/2) \\ \sin(\omega_2 t/2) & -\cos(\omega_2 t/2) \end{pmatrix}, \qquad (26)$$

then in accordance with (8) the time-independent Hamiltonian (10) turns to the time-dependent Hamiltonian $H(t)$ in the form

$$H(r,t) = \left(p_r^2 + q(r)\right)\hat{I} +$$

$$\begin{pmatrix} \frac{V_{11}-V_{22}}{2}\cos\omega_2 t - V_{12}\sin\omega_2 t & V_{12}\cos\omega_2 t + \frac{V_{11}-V_{22}}{2}\sin\omega_2 t - i\omega_2/2 \\ V_{21}\cos\omega_2 t + \frac{V_{11}-V_{22}}{2}\sin\omega_2 t + i\omega_2/2 & -\frac{V_{11}-V_{22}}{2}\cos\omega_2 t + V_{12}\sin\omega_2 t \end{pmatrix} \qquad (27)$$

In accordance with (18) $\left|\Psi_v\right\rangle = S(t)\exp\left(-i\tilde{E}_v t\right)\left|\Phi_v(r)\right\rangle$ the recurrent solutions are immediately written as

$$\left|\Psi_v(r,t)\right\rangle = \begin{pmatrix} \cos(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_1^v t) & -\sin(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_2^v t) \\ \sin(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_1^v t) & \cos(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_2^v t) \end{pmatrix}\left|\Phi_v(r)\right\rangle \qquad (28)$$

and in one period

$$\left|\Psi_v(r,(T+t))\right\rangle = -\begin{pmatrix} \exp\left(-i\tilde{\varepsilon}_1^v T\right) & 0 \\ 0 & \exp\left(-i\tilde{\varepsilon}_2^v T\right) \end{pmatrix}\left|\Psi_v(r,t)\right\rangle$$

When there is no dependence on the space variable, the time-dependent Hamiltonian and the corresponding cyclic solutions can be obtained by applying the transformation $\mathbf{S}_2(\mathbf{t})$ to the space-uniform time-independent Hamiltonian $\tilde{\mathbf{H}} = \boldsymbol{\sigma} \cdot \tilde{\mathbf{B}} = \tilde{\boldsymbol{\Omega}} \begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ \sin\tilde{\theta} & -\cos\tilde{\theta} \end{pmatrix}$

$$\mathbf{H(t)} = \tilde{\Omega} \begin{pmatrix} \cos\tilde{\theta}\cos\omega_2 t - \sin\tilde{\theta}\sin\omega_2 t & \sin\tilde{\theta}\cos\omega_2 t + \cos\tilde{\theta}\sin\omega_2 t - i\omega_2/2\tilde{\Omega} \\ \sin\tilde{\theta}\cos\omega_2 t + \cos\tilde{\theta}\sin\omega_2 t + i\omega_2/2\tilde{\Omega} & -(\cos\tilde{\theta}\cos\omega_2 t - \sin\tilde{\theta}\sin\omega_2 t) \end{pmatrix}$$

$$|\Psi_1(t)\rangle = \begin{pmatrix} \cos(\omega_2 t/2)\exp(-i\tilde{\Omega}t)\cos(\tilde{\theta}/2) - \sin(\omega_2 t/2)\exp(i\tilde{\Omega}t)\sin(\tilde{\theta}/2) \\ \sin(\omega_2 t/2)\exp(-i\tilde{\Omega}t)\cos(\tilde{\theta}/2) + \cos(\omega_2 t/2)\exp(i\tilde{\Omega}t)\sin(\tilde{\theta}/2) \end{pmatrix} \tag{29}$$

$$|\Psi_2(t)\rangle = \begin{pmatrix} -\cos(\omega_2 t/2)\exp(-i\tilde{\Omega}t)\sin(\tilde{\theta}/2) - \sin(\omega_2 t/2)\exp(i\tilde{\Omega}t)\cos(\tilde{\theta}/2) \\ -\sin(\omega_2 t/2)\exp(-i\tilde{\Omega}t)\sin(\tilde{\theta}/2) + \cos(\omega_2 t/2)\exp(i\tilde{\Omega}t)\cos(\tilde{\theta}/2) \end{pmatrix}$$

This problem is identical to the motion of a spin-1/2 in the uniformly rotating magnetic field in the ***x-z*** plane.

3 case. *Let* $\mathbf{S(t)}$ *be chosen as an operator of rotation around x-axis*

$$\mathbf{S_1(t) = \exp(-i\hat{\sigma}_1\omega_1 t/2) = \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix}} \tag{30}$$

Then, from (8) the time-dependent Hamiltonian $\mathbf{H(t)}$ is

$$\mathbf{H(r,t) = \left(p_r^2 + q(r)\right)\hat{I} + \begin{pmatrix} \frac{V_{11}-V_{22}}{2}\cos(\omega_1 t) & V_{12} - \frac{\omega_1}{2} + i\frac{V_{11}-V_{22}}{2}\sin(\omega_1 t) \\ V_{12} - \frac{\omega_1}{2} - i\frac{V_{11}-V_{22}}{2}\sin(\omega_1 t) & -\frac{V_{11}-V_{22}}{2}\cos(\omega_1 t) \end{pmatrix}}. \tag{31}$$

By analogy with the previous cases, the Hamiltonian (31) can be represented in the form for a spin-1/2 particle in the space-inhomogeneous magnetic field precessing around 0x axis:

$$\mathbf{H(r,t) = \left(p_r^2 + q(r)\right)\hat{I} + \tilde{\Omega}(r)\begin{pmatrix} \cos\tilde{\theta}(r)\cos(\omega_1 t) & \sin\tilde{\theta}(r) - \frac{\omega_1}{2\Omega(r)} + i\cos\tilde{\theta}(r)\sin\omega_1 t \\ \sin\tilde{\theta}(r) - \frac{\omega_1}{2\Omega(r)} - i\cos\theta(r)\sin\omega_1 t & -\cos\tilde{\theta}(r)\cos(\omega_1 t) \end{pmatrix}} \tag{32}$$

The cyclic solutions with account of $\mathbf{|\Psi_\nu\rangle = S(t)\exp\left(-i\tilde{E}_\nu t\right)|\Phi_\nu(r)\rangle}$ are written as

$$\mathbf{|\Psi_\nu(r,t)\rangle = \begin{pmatrix} \cos(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_1^\nu t) & -i\sin(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_2^\nu t) \\ -i\sin(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_1^\nu t) & \cos(\omega_1 t/2)\exp(-i\tilde{\varepsilon}_2^\nu t) \end{pmatrix}|\Phi_\nu(r)\rangle} \tag{33}$$

When there is no dependence on the space variable, the time-dependent Hamiltonian and corresponding cyclic solutions can be obtained by applying the transformation $\mathbf{S_1(t) = \exp(-i\hat{\sigma}_1\omega_1 t/2)}$ to the space-uniform time-independent Hamiltonian (22)

$$\mathbf{H(t) = \tilde{\Omega}\begin{pmatrix} \cos\tilde{\theta}\cos(\omega_1 t) & \sin\tilde{\theta} - \dfrac{\omega_1}{2\tilde{\Omega}} + i\cos\tilde{\theta}\sin(\omega_1 t) \\ \sin\tilde{\theta} - \dfrac{\omega_1}{2\tilde{\Omega}} - i\cos\tilde{\theta}\sin(\omega_1 t) & -\cos\tilde{\theta}\cos(\omega_1 t) \end{pmatrix}} \qquad (34)$$

$$\mathbf{\left|\Psi_1(t)\right\rangle = \begin{pmatrix} \cos(\omega_1 t/2)\exp(-i\tilde{\Omega}t)\cos(\tilde{\theta}/2) - i\sin(\omega_1 t/2)\exp(i\tilde{\Omega}t)\sin\left(\tilde{\theta}/2\right) \\ -i\sin(\omega_1 t/2)\exp(-i\tilde{\Omega}t)\cos(\tilde{\theta}/2) + \cos(\omega_1 t/2)\exp(i\tilde{\Omega}t)\sin(\tilde{\theta}/2) \end{pmatrix}}$$

$$\mathbf{\left|\Psi_2(t)\right\rangle = \begin{pmatrix} -\cos(\omega_1 t/2)\exp(-i\tilde{\Omega}t)\sin(\tilde{\theta}/2) - i\sin(\omega_1 t/2)\exp(i\tilde{\Omega}t)\cos\left(\tilde{\theta}/2\right) \\ -i\sin(\omega_1 t/2)\exp(-i\tilde{\Omega}t)\sin(\tilde{\theta}/2) + \cos(\omega_1 t/2)\exp(i\tilde{\Omega}t)\cos(\tilde{\theta}/2) \end{pmatrix}}$$

Thus, by a different choice of transformation operators $\mathbf{S_i(t), i = 1,2,3}$, three families of time-dependent potential matrices with the corresponding cyclic solutions are generated in a closed form from one family of time-independent potential matrices.

It is evident, in all considered cases the Hamiltonians are $\mathbf{T}$ - *periodic,* $T = 2\pi/\omega$

1)

$$\mathbf{H}(\mathbf{r},\mathbf{t}) = \left(\mathbf{p}_r^2 + \mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}} + \begin{pmatrix} \dfrac{\mathbf{V}_{11}(\mathbf{r}) - \mathbf{V}_{22}(\mathbf{r}) + \boldsymbol{\omega}}{\mathbf{2}} & \mathbf{V}_{12}(\mathbf{r})\exp\left(-\mathbf{i}\boldsymbol{\omega}\mathbf{t}\right) \\ \mathbf{V}_{21}(\mathbf{r})\exp\left(\mathbf{i}\boldsymbol{\omega}\mathbf{t}\right) & -\dfrac{\mathbf{V}_{11}(\mathbf{r}) - \mathbf{V}_{22}(\mathbf{r}) + \boldsymbol{\omega}}{\mathbf{2}} \end{pmatrix}$$

2)

$$\mathbf{H}(\mathbf{r},\mathbf{t}) = \left(\mathbf{p}_r^2 + \mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}} + \begin{pmatrix} \dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{cos}(\boldsymbol{\omega}_2\mathbf{t}) - \mathbf{V}_{12}\mathbf{sin}(\boldsymbol{\omega}_2\mathbf{t}) & \mathbf{V}_{12}\mathbf{cos}(\boldsymbol{\omega}_2\mathbf{t}) + \dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{sin}(\boldsymbol{\omega}_2\mathbf{t}) - \mathbf{i}\boldsymbol{\omega}_2/\mathbf{2} \\ \mathbf{V}_{21}\mathbf{cos}(\boldsymbol{\omega}_2\mathbf{t}) + \dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{sin}(\boldsymbol{\omega}_2\mathbf{t}) + \mathbf{i}\boldsymbol{\omega}_2/\mathbf{2} & -\dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{cos}(\boldsymbol{\omega}_2\mathbf{t}) + \mathbf{V}_{12}\mathbf{sin}(\boldsymbol{\omega}_2\mathbf{t}) \end{pmatrix}$$

3)

$$\mathbf{H}(\mathbf{r},\mathbf{t}) = \left(\mathbf{p}_r^2 + \mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}} + \begin{pmatrix} \dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{cos}(\boldsymbol{\omega}_1\mathbf{t}) & \mathbf{V}_{12} - \dfrac{\boldsymbol{\omega}_1}{\mathbf{2}} + \mathbf{i}\dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{sin}(\boldsymbol{\omega}_1\mathbf{t}) \\ \mathbf{V}_{12} - \dfrac{\boldsymbol{\omega}_1}{\mathbf{2}} - \mathbf{i}\dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{sin}(\boldsymbol{\omega}_1\mathbf{t}) & -\dfrac{\mathbf{V}_{11} - \mathbf{V}_{22}}{\mathbf{2}}\mathbf{cos}(\boldsymbol{\omega}_1\mathbf{t}) \end{pmatrix}$$

They are obtained from, $\mathbf{H}(\mathbf{r}) = \left(\mathbf{p}_r^2 + \mathbf{q}(\mathbf{r})\right)\hat{\mathbf{I}} + \begin{pmatrix} \dfrac{\mathbf{V}_{11}(\mathbf{r}) - \mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}} & \mathbf{V}_{12}(\mathbf{r}) \\ \mathbf{V}_{21}(\mathbf{r}) & -\dfrac{\mathbf{V}_{11}(\mathbf{r}) - \mathbf{V}_{22}(\mathbf{r})}{\mathbf{2}} \end{pmatrix}$

the corresponding solutions

1)

$$\left|\boldsymbol{\Psi}_{\nu}(\mathbf{r},\mathbf{t})\right\rangle = \begin{pmatrix} \exp\left(-\mathbf{i}\left(\tilde{\varepsilon}_1^{\nu}+\omega/2\right)\mathbf{t}\right) & \mathbf{0} \\ \mathbf{0} & \exp\left(-\mathbf{i}\left(\tilde{\varepsilon}_1^{\nu}-\omega/2\right)\mathbf{t}\right) \end{pmatrix} \left|\boldsymbol{\Phi}_{\nu}(\mathbf{r})\right\rangle$$

2)

$$\left|\boldsymbol{\Psi}_{\nu}(\mathbf{r},\mathbf{t})\right\rangle = \begin{pmatrix} \cos\left(\omega_2\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_1^{\nu}\mathbf{t}) & -\sin\left(\omega_2\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_2^{\nu}\mathbf{t}) \\ \sin\left(\omega_2\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_1^{\nu}\mathbf{t}) & \cos\left(\omega_2\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_2^{\nu}\mathbf{t}) \end{pmatrix} \left|\boldsymbol{\Phi}_{\nu}(\mathbf{r})\right\rangle$$

3)

$$\left|\boldsymbol{\Psi}_{\nu}(\mathbf{r},\mathbf{t})\right\rangle = \begin{pmatrix} \cos\left(\omega_1\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_1^{\nu}\mathbf{t}) & -\mathbf{i}\sin\left(\omega_1\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_2^{\nu}\mathbf{t}) \\ -\mathbf{i}\sin\left(\omega_1\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_1^{\nu}\mathbf{t}) & \cos\left(\omega_1\mathbf{t}/2\right)\exp(-\mathbf{i}\tilde{\varepsilon}_2^{\nu}\mathbf{t}) \end{pmatrix} \left|\boldsymbol{\Phi}_{\nu}(\mathbf{r})\right\rangle$$

are $2T = 4\pi/\omega$ *-periodic and change their sign after one period*

Clearly, a more general transformations can be taken: i) as a direct product of the $\mathbf{S}_{\mathrm{i}}(\mathbf{t})$***,*** $\mathbf{S}(\mathbf{t}) = \prod_{\mathbf{i}=1}^{3}\mathbf{S}_{\mathrm{i}}(\mathbf{t})$***, ii) with a more complicated dependence of*** S ***on time. Whenever*** $V(r)$ ***is an exactly soluble potential matrix for the ordinary time-independent system of Schr. Eqs., a family of exactly soluble time-dependent Hamiltonians of the Schr. Eqs. can be generated***

Time-dependent exactly solvable models for quantum computing

- **A time-dependent periodic Hamiltonian admitting exact solutions is applied to construct a set of universal gates for quantum computer.**
- **The time evolution matrices are obtained in an explicit form and used to construct logic gates for computation. A way of obtaining entanglement operator is discussed.**

A quantum computer is composed of a set of qubits which can be manipulated in a controlled way. Any quantum two-level systems can be taken to create qubits. *A computation process corresponds to the evolution of the set of the qubits according to a specific unitary operator, for example evolution operator* $\mathbf{U(t)}$. A general operation is decomposed into a discrete sequence in time of operations - quantum gates.

The physical realization of the qubit register and a universal set of one-qubit and two-qubit logic gates is an important problem of quantum computation. Here we shall construct one-qubit and two-cubit gates with desired properties controlled by time-dependent Hamiltonian.

The simplest unit of quantum information is a quantum bit, or ***qubit***. The qubit is a vector in a two-dimensional Hilbert space, which can be presented as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The basis vectors $|0\rangle$ and $|1\rangle$ are chosen as $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$

Here α and β are complex coefficients which satisfy the condition $|\alpha|^2 + |\beta|^2 = 1$. Then $|\psi\rangle$ is the normalized vector, and α^2 and β^2 characterize the probabilities of the results $|0\rangle$ and $|1\rangle$, correspondingly. The 2-nd order matrices $U(2\times 2)$ transform one-qubit states and describe an evolution their in time:

$$|\psi_f\rangle = U(2\times 2)|\psi_0\rangle \qquad U(t) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Such transformations in quantum computation determine one-qubit quantum operations - ***quantum gates.***

The formalism of quantum mechanics is usually applied not to individual systems but to ensembles of systems. In quantum computation, the state of the computer of *n* qubits can be expressed as a vector $|\boldsymbol{\Psi}\rangle$ in a space of dimension 2^n. Vector $|\boldsymbol{\Psi}\rangle$ of the quantum register from *n* qubits is expressed as a complex linear superpozition of 2^n basis states

$$|\boldsymbol{\Psi}\rangle = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{2^n-1}} \mathbf{a_k} |\mathbf{j_k}\rangle \qquad (35)$$

Here a_k are projections of the vector $|\Psi\rangle$ on the directions of basis states $|\mathbf{j_0}\rangle$, $|\mathbf{j_1}\rangle$,... $|\mathbf{j_{2^n-1}}\rangle$, $\sum_k a_k^2 = 1$. Basis states $|\mathbf{j}\rangle = |\mathbf{i_1}, \mathbf{i_2} \ldots \mathbf{i_n}\rangle = |\mathbf{i_1}\rangle \otimes |\mathbf{i_2}\rangle \otimes \ldots |\mathbf{i_n}\rangle$, $\mathbf{i_1}, \mathbf{i_2} \ldots \mathbf{i_n} = \mathbf{or}\, |\mathbf{0}\rangle \mathbf{or}\, |\mathbf{1}\rangle$ are presented as

$$\begin{gathered} |\mathbf{j_0}\rangle = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \ldots |\mathbf{0}\rangle \\ |\mathbf{j_1}\rangle = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \ldots |\mathbf{1}\rangle \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ |\mathbf{j_{2^n-1}}\rangle = |\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes \ldots |\mathbf{1}\rangle \end{gathered} \qquad (36)$$

The transformation of an initial state vector $|\boldsymbol{\Psi}_\mathbf{0}\rangle$ *into the final vector* $|\boldsymbol{\Psi}_\mathbf{f}\rangle$ *models the process of calculation on quantum computer*

$$|\boldsymbol{\psi}_\mathbf{f}\rangle = \mathbf{U}(\mathbf{2^n \times 2^n})|\boldsymbol{\psi}_\mathbf{0}\rangle$$

Vectors $|\Psi_0\rangle$ and $|\Psi_f\rangle$ are vectors in the 2^n Hilbert space.

The transformation matrices $\mathbf{U}\left(\mathbf{2^n \times 2^n; t}\right)$ define the dynamic evolution of the quantum system from *n* qubits. At the same time, the matrices $\mathbf{U}\left(\mathbf{2^n \times 2^n; t}\right)$ provide the process of quantum computing at each fixed moment.

Clearly, the realization of the transformation $\mathbf{U}\left(\mathbf{2^n \times 2^n}\right)$ with $n>3$ is a very difficult problem. As usual, *Bryglinski J.L., Bryglinski R. "Universal Quantum Gates in Mathematics of Quantum Computation", Chapman and Hall/CRC Press, Florida, 2002*, one considers the presentation of $\mathbf{U}\left(\mathbf{2^n \times 2^n}\right)$ as a production of second $\mathbf{U}(\mathbf{2 \times 2})$ order and forth $\mathbf{U}(\mathbf{4 \times 4})$ order matrices

$$\mathbf{U}\left(\mathbf{2^n \times 2^n}\right) = \prod_{\mathbf{i,j}} \mathbf{U_i}(\mathbf{2 \times 2}) \otimes \mathbf{U_j}\left(\mathbf{2^2 \times 2^2}\right) \qquad (37)$$

We shall show how it is possible explicitly to generate one-qubit logic gates, given by $\mathbf{2 \times 2}$ ***unitary operators, from the time evolution matrices and give a way of obtaining entanglement*** $\mathbf{4 \times 4}$ ***operators***.

A universal gate set

The universal 1-qubit logic gates can be constructed from the time evolution matrices which we obtain in a closed analytic form. In our approach, the time-dependent periodic Hamiltonians admitting exact solutions are applied to control the time evolution of the 1-qubit gates.

Suppose that the time evolution of the quantum system is governed by the Schrödinger Eq.

$$\mathbf{i}\frac{\mathbf{d}|\boldsymbol{\Psi}(\mathbf{r},\mathbf{t})\rangle}{\mathbf{dt}} = \mathbf{H}(\mathbf{r},\mathbf{t})|\boldsymbol{\Psi}(\mathbf{r},\mathbf{t})\rangle \qquad (38)$$

with **T** - periodic time-dependent Hamiltonian, $\mathbf{H(t)} = \mathbf{H(t+T)}$.

Assume, the initial state of the qubit can be written in one of the states of the time independent Hamiltonian $\tilde{\mathbf{H}}$:

$$\tilde{\mathbf{H}} = \boldsymbol{\sigma}\cdot\tilde{\mathbf{B}} = \tilde{\boldsymbol{\Omega}}\begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ \sin\tilde{\theta} & -\cos\tilde{\theta} \end{pmatrix} \qquad (39)$$

$$\varphi_1 = \cos\tilde{\theta}/2|0\rangle + \sin\tilde{\theta}/2|1\rangle \qquad \text{or} \qquad \varphi_2 = -\sin\tilde{\theta}/2|0\rangle + \cos\tilde{\theta}/2|1\rangle$$

Taking the gauge transformation as $|\boldsymbol{\Psi}(\mathbf{r},\mathbf{t})\rangle = \mathbf{S(t)}|\boldsymbol{\Phi}(\mathbf{r},\mathbf{t})\rangle \quad \mathbf{S(t)} = \exp(-\mathbf{i}\boldsymbol{\sigma}_x\boldsymbol{\omega}_1\mathbf{t}/2)$ (40)

the time-independent Hamiltonian (39) is changed to the time-dependent one:

$$\mathbf{H(t)} = \mathbf{S(t)}\tilde{\mathbf{H}}\mathbf{S}^+(\mathbf{t}) + \mathbf{i}\dot{\mathbf{S}}(\mathbf{t})\mathbf{S}^+(\mathbf{t}) \qquad (41)$$

The evolution operator $\mathbf{U(t)=\exp\left(-i\sigma_x\omega_1 t/2\right)\exp\left(-i\widetilde{H}t\right)}$, corresponding to the time-dependent Hamiltonian

$$\mathbf{H(t)=\lambda\begin{pmatrix}\cos\widetilde{\theta}\cos(\omega_1 t) & \sin\widetilde{\theta}-\frac{\omega_1}{2\lambda}+i\cos\widetilde{\theta}\sin\omega_1 t\\ \sin\widetilde{\theta}-\frac{\omega_1}{2\lambda}-i\cos\widetilde{\theta}\sin\omega_1 t & -\cos\widetilde{\theta}\cos(\omega_1 t)\end{pmatrix}} \qquad (41^*)\,.$$

is written as

$$\mathbf{U_1(t)=\begin{pmatrix}\cos\left(\omega_1 t/2\right) & -i\sin\left(\omega_1 t/2\right)\\ -i\sin\left(\omega_1 t/2\right) & \cos\left(\omega_1 t/2\right)\end{pmatrix}\begin{pmatrix}\exp(-i\lambda t) & 0\\ 0 & \exp(i\lambda t)\end{pmatrix}} \qquad (42)$$

The time evolution matrix ***U(t)*** gives the set of the universal one-qubit gates, which are controlled by the parameters $\boldsymbol{\omega_1}$ and $\boldsymbol{\lambda}$.

An important one-bit transformation is the operation of negation or inversion operation $\mathbf{NOT=\sigma_x}$. The gate $\mathbf{NOT}$ can be obtained from (42) at $\boldsymbol{\omega_1 t=\pi}$ and $\boldsymbol{\lambda t=2n\pi}$ and then after multiplication of the result by ***i*:**

$$\mathbf{NOT=iU_1\left(\omega_1 t=\pi,\lambda t=2n\pi\right)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}} \qquad (43)$$

The transformation $\mathbf{NOT}$ exchanges $\left|\mathbf{0}\right\rangle$ and $\left|\mathbf{1}\right\rangle$, e.g. $\mathbf{NOT\left(a\left|0\right\rangle+b\left|1\right\rangle\right)=a\left|1\right\rangle+b\left|0\right\rangle}$

Another special one qubit gate can be obtained from (42) at $\boldsymbol{\omega_1 t = \pi}$ and $\boldsymbol{\lambda t = \pi/2}$ and after multiplication of the result by ***i***:

$$\mathbf{Y} = \mathbf{iU_1}(\boldsymbol{\omega_1}\mathbf{t} = \boldsymbol{\pi}, \boldsymbol{\lambda}\mathbf{t} = \boldsymbol{\pi}/\mathbf{2}) = \begin{pmatrix} \mathbf{0} & -\mathbf{i} \\ \mathbf{i} & \mathbf{0} \end{pmatrix} = \boldsymbol{\sigma}_{\mathrm{y}} \tag{44}$$

The special gate Z is obtained from (42) at $\boldsymbol{\omega_1 t = 4\pi}$ and $\boldsymbol{\lambda t = \pi/2}$ and after multiplication by *i*:

$$\mathbf{Z} = \mathbf{iU_1}(\boldsymbol{\omega_1}\mathbf{t} = \mathbf{4}\boldsymbol{\pi}, \boldsymbol{\lambda}\mathbf{t} = \boldsymbol{\pi}/\mathbf{2}) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = \boldsymbol{\sigma}_{\mathrm{z}} \tag{45}$$

Now let us obtain another important one-bit transformation. It is the *Hadamar transformation* defined by

$$\mathbf{H} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\boldsymbol{\sigma}_{\mathrm{x}} + \boldsymbol{\sigma}_{\mathrm{z}}) \tag{46}$$

When applied to $|\mathbf{0}\rangle$ and to $|\mathbf{1}\rangle$, the Hadamar gate **H** creates to the superposition of states with the equal probability

$$\mathbf{H}|\mathbf{0}\rangle = \mathbf{H}\begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|\mathbf{0}\rangle + |\mathbf{1}\rangle)$$

$$\mathbf{H}|\mathbf{1}\rangle = \mathbf{H}\begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(|\mathbf{0}\rangle - |\mathbf{1}\rangle)$$

If the initial state of the qubit is $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, then the evolution matrix ***U(t)*** corresponding to the time-dependent Hamiltonian (41^*) is written as

$$\mathbf{U(t) = \exp(-\sigma_x \omega_1 t/2)\exp(-i\sigma_z \lambda t)\exp(-i\sigma_y \tilde{\theta}/2) =}$$

$$= \begin{pmatrix} \cos(\omega_1 t/2) & -i\sin(\omega_1 t/2) \\ -i\sin(\omega_1 t/2) & \cos(\omega_1 t/2) \end{pmatrix} \begin{pmatrix} \exp(-i\lambda t) & 0 \\ 0 & \exp(i\lambda t) \end{pmatrix} \begin{pmatrix} \cos(\tilde{\theta}/2) & -\sin(\tilde{\theta}/2) \\ \sin(\tilde{\theta}/2) & \cos(\tilde{\theta}/2) \end{pmatrix} \quad (47)$$

At **t=0**, $\tilde{\theta} = \pi/2$ and any ω_1 , λ from (47) we obtain the gate

$$\mathbf{U(\omega_1, \lambda; t = 0, \tilde{\theta} = \pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}} \quad (48)$$

To obtain the Hadamar gate, we multiply $\mathbf{NOT = iU_1(\pi, 2\pi n)}$ on the gate **U.** Therefore, the Hadamar gate **H** is a result of the sequence of two transformations

$$\mathbf{H = iU_1(\pi, 2\pi n) U(\omega_1, \lambda; t = 0, \tilde{\theta} = \pi/2)} \quad (49)$$

Note $U_1(t) = U(t;\theta = 0)$. Applied to ***n*** bits, **H** generates superposition of all $\mathbf{2^n}$ possible states, which can be considered as a binary representation of the numbers from 0 to $\mathbf{2^n}$**-1**

$$\mathbf{(H \otimes H \otimes ... \otimes H)|00...0\rangle = \frac{1}{\sqrt{2^n}}((|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes ... \otimes (|0\rangle + |1\rangle)) = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} |j_k\rangle} \quad (50)$$

It should be noted, the presented above sequence of gates for obtaining states $\frac{1}{\sqrt{2^n}}\sum_{k=0}^{2^n-1}|j_k\rangle$ are not unique. given examples can be used for building one qubit gates.

Construction of two-qubit gates

The 2-nd order matrices $U_i(2\times 2)$ transform one-qubit states. The 4-th order matrices $U_i(2^2\times 2^2)$ transform couples of one-qubit states. There are 4 basis states in 4-th dimension Hilbert space for two-cubit systems building on one-cubit states $|0\rangle$, $|1\rangle$:

$$|00\rangle = |0\rangle\otimes|0\rangle, |01\rangle = |0\rangle\otimes|1\rangle, |10\rangle = |1\rangle\otimes|0\rangle, |11\rangle = |1\rangle\otimes|1\rangle$$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Any two-qubit state can be expressed as a superposition of these basis states

$$|\Psi\rangle = c_{00}|00\rangle + c_{10}|10\rangle + c_{01}|01\rangle + c_{11}|11\rangle, \qquad (51)$$

where $|c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2 = 1$.

Entanglement

A gate *G* is said to be entangling, if $|\mathbf{\Psi}\rangle = \mathbf{G}|\psi_1\rangle \otimes |\psi_2\rangle$ is not decomposable as a tensor product of two one-qubit states. It means that $|\mathbf{\Psi}_{12}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$. *The property* $|\mathbf{\Psi}_{12}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$ *is called* ***entanglement****. If in (51)* $\mathbf{c}_{00}\mathbf{c}_{11} - \mathbf{c}_{01}\mathbf{c}_{10} \neq \mathbf{0}$*, then* $|\mathbf{\Psi}\rangle$ ***is an entangled state****.*

In our case the entanglement operator is obtained from two independent systems with the use of unitary gauge time-dependent transformations, which lead to time-dependent periodic operators and entanglement of states.

Construction of the Hamiltonian with the desired entangled operator

Let **H** is a $\mathbf{2 \times 2}$ Hamiltonian in the form

$$\mathbf{H} = \mathbf{h} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{h} + \boldsymbol{\varepsilon}\mathbf{A} \qquad (52)$$

where $\varepsilon \in \{0,1\}$ and *h* is a diagonal time-independent Hamiltonian

$$\mathbf{h} = \begin{pmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{b} \end{pmatrix}.$$

The evolution operator of the matrix Schrödinger Eq. $\mathbf{i}\frac{\mathbf{d}|\mathbf{\Psi}\rangle}{\mathbf{dt}} = \mathbf{H}|\mathbf{\Psi}\rangle$ with the Hamiltonian (52) is expressed as follows

$$\mathbf{U}(\mathbf{t}) = \left(\mathbf{exp}^{-\mathbf{iht}} \otimes \mathbf{exp}^{-\mathbf{iht}}\right)\mathbf{exp}^{-\mathbf{iAt}}$$

if the operator ***A*** commutes with the Hamiltonian $\mathbf{h} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{h}$.

We would like to get the entanglement operator **U(t)** and to construct a corresponding Hamiltonian in the form (52). To this end, let us select the operator $\mathbf{R(t)=e^{-iAt}}$ in the form

$$\mathbf{R(t)=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos(t) & -i\sin(t) & 0\\ 0 & -i\sin(t) & \cos(t) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}} \qquad (53)$$

Find **A(t)** form

$$\mathbf{A=i\frac{dR(t)}{dt}R^{-1}(t)=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}} \qquad (54)$$

The matrix

$$\mathbf{h=\frac{\sigma_3}{2}=\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}$$

satisfies the condition of commutation $\mathbf{[A,(h\otimes 1+1\otimes h)]}$.

At last, substitution $\mathbf{e}^{-\mathbf{iAt}}$ and ***h*** into the evolution matrix $\mathbf{U(t)} = \left(\mathbf{e}^{-\mathbf{iht}} \otimes \mathbf{e}^{-\mathbf{iht}}\right)\mathbf{e}^{-\mathbf{iAt}}$ gives the entanglement operator

$$\mathbf{U(t)} = \begin{pmatrix} \mathbf{e}^{\mathbf{it}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{\cos(t)} & -\mathbf{i\sin(t)} & \mathbf{0} \\ \mathbf{0} & -\mathbf{i\sin(t)} & \mathbf{\cos(t)} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{e}^{-\mathbf{it}} \end{pmatrix}$$

So, the entanglement operator has been obtained with the use of the unitary time-dependent transformation $\mathbf{R(t)} = \mathbf{e}^{-\mathbf{iAt}}$ (53), which leads to the time-dependent periodic operator **U(t)** and entanglement of states. We obtain the corresponding Hamiltonian (52) with ***A*** as given in (54).

Conclusion

- Suggested method allows to generate exactly soluble time-dependent Hamiltonians from time-independent ones.

- The time evolution matrices are obtained in an explicit form and used to construct logic gates for computation.

- The approach opens opportunities for modelling quantum dynamic systems with predetermined properties, in particular, quantum wells with properties of dynamic localization.

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Thank you for attention