

Hydrogenlike systems in Poincare-covariant model

Viktor Andreev

Department of Theoretical Physics, Gomel State University
Gomel, Belarus

Abstract

We consider relativistic formulation of Hydrogenlike systems based on Poincare-covariant Hamiltonian dynamics. The kernel of radial equation for relativistic Hydrogenlike system is calculated exactly in momentum space. The semi-spectral Chebyshev method of numerical calculation can be used for solving of integral equations of relativistic bound state with high precision (25 – 60 Hz for $n = 1$ and 100 kHz for $n = 5$). We discuss some relativistic effects of the one-loop and triangle corrections to the splitting interval of Hydrogenlike systems.

Introduction

The investigation of the energy spectra of hydrogenic atoms is of great importance for high accuracy verification of the Standard Model and derivation more correct values of fundamental physical constants (the fine structure constant, the masses of the muon and electron, the proton charge radius etc.) *Eides:2000, Karshenboim:2003, Karshenboim:2006*. The important stimulus for these evaluation is provided by the spectacular experimental progress in measurements of two fermion system energy levels. The relative uncertainty of measurement of the frequency of atomic energy levels was reduced to $3,4 \cdot 10^{-13}$ (or ~ 60 kHz) *Eides:2000, Karshenboim:2003*. So the important aim is to investigate possibilities of calculation eigenvalues for fermion-fermion system with high precision.

Our approach includes: 1) The spinor part of radial equation for relativistic Hydrogenlike system is calculated exactly in momentum space 2) For

calculation eigenvalues of relativistic Hydrogenlike system we use method in which integral equation are solved numerically by means meshing of the semi-spectral Chebyshev method *Deloff:2006*.

For calculation of different energy levels of fermion-fermion systems with electromagnetic interaction there exist some models. Model based on effective Dirac equation is most popular *Eides:2000*. Quasipotential approach for description bound Hydrogenlike atoms was realized in Refs *Faustov:1997, Martynenko:2006, Skacnkov2001*. In our article we use the description of Hydrogenlike systems with the help of the relativistic Hamiltonian dynamics (RHD) (or relativistic quantum mechanics) *Keister:1991*.

In this approach, the Hamiltonian \hat{H} is assumed to be the sum of a relativistic kinetic energy operator $T(\mathbf{k})$ that represents the invariant mass of two noninteracting particles plus phenomenological interaction \hat{V} . The kinetic

energy operator has the form

$$T(\mathbf{k}) = \sqrt{m_1^2 + \mathbf{k}^2} + \sqrt{m_2^2 + \mathbf{k}^2}, \quad (1)$$

where

$$\mathbf{k} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2) + \frac{\mathbf{P}}{M_0} \left(\frac{m_2^2 - m_1^2 - M_0 [\omega_{m_2}(\mathbf{p}_2) - \omega_{m_1}(\mathbf{p}_1)]}{\omega_{M_0}(\mathbf{P}) + M_0} \right) \quad (2)$$

is the relative momentum. In Eq.(2) the momentum \mathbf{P} is the total momentum of the free-system

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad (3)$$

and $M_0 = T(\mathbf{k}) = \omega_{m_1}(\mathbf{k}) + \omega_{m_2}(\mathbf{k})$, $\omega_m(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}$, $k = |\mathbf{k}|$.

The eigenvalue problem for the mass of a bound system Ψ with momentum \mathbf{Q} , spin J and spin's projection μ can be written as follow:

$$\hat{M} | \Psi_{\mathbf{Q}, J, \mu} \rangle \equiv \left(M_0 + \hat{V} \right) | \Psi_{\mathbf{Q}, J, \mu} \rangle = E | \Psi_{\mathbf{Q}, J, \mu} \rangle. \quad (4)$$

The eigenvalues of four-momenta and spin variables have different kinds

depending on the forms of the RHD. The state vectors are diagonal on their eigenvalues of operators that coincide for both free and bound systems (so called kinematic set). In instant form of RHD the wave functions are

$$\langle \mathbf{p}_1, \lambda_1; \mathbf{p}_2, \lambda_2 | \Psi_{\mathbf{Q}, J, \mu} \rangle = \delta(\mathbf{Q} - \mathbf{p}_1 - \mathbf{p}_2) \Psi_{\mathbf{Q}; \lambda_1 \lambda_2}^{J\mu}(\mathbf{p}_1, \mathbf{p}_2) \quad (5)$$

are determined through the basis of direct product $|\mathbf{p}_1, \lambda_1; \mathbf{p}_2, \lambda_2\rangle$ or wave functions

$$\langle \mathbf{P}, \mathbf{k}, \lambda_1, \lambda_2 | \Psi_{\mathbf{Q}, J, \mu} \rangle = \delta(\mathbf{P} - \mathbf{Q}) \Phi_{\mathbf{Q}; \lambda_1, \lambda_2}^{J\mu}(\mathbf{k}), \quad (6)$$

are determined through the basis

$$|\mathbf{P}, \mathbf{k}, \lambda_1, \lambda_2\rangle = \sqrt{\frac{\omega_{m_1}(\mathbf{p}_1) \omega_{m_2}(\mathbf{p}_2) M_0}{\omega_{m_1}(\mathbf{k}) \omega_{m_2}(\mathbf{k}) \omega_{M_0}(\mathbf{P})}} |\mathbf{p}_1, \lambda_1, \mathbf{p}_2, \lambda_2\rangle. \quad (7)$$

Hence in this approach the bound system with the momentum \mathbf{Q} , eigenvalues E and spin J is described by the wave function (6) of two-particle state,

which satisfies the equation

$$\begin{aligned} \sum_{\lambda_1, \lambda_2} \int \langle \mathbf{k}, \sigma_1, \sigma_2 \parallel \hat{V} \parallel \mathbf{k}', \lambda_1, \lambda_2 \rangle \Phi_{\mathbf{Q}; \lambda_1 \lambda_2}^{J\mu}(\mathbf{k}') d\mathbf{k}' = \\ = (E - T(\mathbf{k})) \Phi_{\mathbf{Q}; \sigma_1 \sigma_2}^{J\mu}(\mathbf{k}) \end{aligned} \quad (8)$$

with reduced matrix element \hat{V}

$$\begin{aligned} \langle \mathbf{P}, \mathbf{k}, \sigma_1, \sigma_2 \parallel \hat{V} \parallel \mathbf{P}', \mathbf{k}', \lambda_1, \lambda_2 \rangle = \\ = \delta(\mathbf{P} - \mathbf{P}') \langle \mathbf{k}, \sigma_1, \sigma_2 \parallel \hat{V} \parallel \mathbf{k}', \lambda_1, \lambda_2 \rangle . \end{aligned} \quad (9)$$

The radial equation for two-particle bound state in the center-momentum system has the following form

$$\sum_{L', S'} \int_0^{\infty} V_{L, S; L', S'}^J(\mathbf{k}, \mathbf{k}') \Phi_{L', S'}^{J\mu}(\mathbf{k}') k'^2 d\mathbf{k}' = (E - M_0) \Phi_{L, S}^{J\mu}(\mathbf{k}) . \quad (10)$$

The equation (10) can be obtained with the help of the Clebsh-Gordan coefficients of Poincaré group. The corresponding potential operator $V_{L',S',L,S}^J(\mathbf{k}', \mathbf{k}) = \langle \mathbf{k}', J, \mu, L', S' \parallel \hat{V} \parallel \mathbf{k}, J, \mu, L, S \rangle$ is obtained from equation

$$V_{L',S',L,S}^J(\mathbf{k}', \mathbf{k}) = \frac{\sqrt{(2L+1)(2L'+1)}}{2J+1} \sum_{\lambda_{1,2}, \lambda'_{1,2}} \left\langle 1/2 \frac{\lambda_1}{2} ; 1/2 - \frac{\lambda_2}{2} \left| S \frac{\lambda}{2} \right\rangle \right. \\ \left. \left\langle L 0 ; S \frac{\lambda}{2} \left| J \frac{\lambda}{2} \right\rangle \left\langle 1/2 \frac{\lambda'_1}{2} ; 1/2 - \frac{\lambda'_2}{2} \left| S \frac{\lambda'}{2} \right\rangle \right. \right. \\ \left. \left. \left\langle L 0 ; S \frac{\lambda'}{2} \left| J \frac{\lambda'}{2} \right\rangle \langle \mathbf{k}', J, \mu, \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, J, \mu, \lambda_1, \lambda_2 \rangle \right. \right. \quad (11)$$

Matrix element $\langle \mathbf{k}', J', \mu', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, J, \mu, \lambda_1, \lambda_2 \rangle$ is related by with $\langle \mathbf{k}', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 \rangle$ by means of the Jacob-Wick decomposition

(see, for example **Brown79**) have the form

$$\begin{aligned} \langle \mathbf{k}', J', \mu', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, J, \mu, \lambda_1, \lambda_2 \rangle = & \frac{\sqrt{(2J+1)(2J'+1)}}{4\pi} \\ & \int d^2\hat{\mathbf{k}} d^2\hat{\mathbf{k}}' D_{\mu' \lambda'}^{J'}(\varphi_{k'}, \theta_{k'}, -\varphi_{k'}) D_{\mu \lambda}^{*J}(\varphi_k, \theta_k, -\varphi_k) \\ & \langle \mathbf{k}', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 \rangle, \end{aligned} \quad (12)$$

with $\lambda = (\lambda_1 - \lambda_2) / 2$. Hence $\langle s_1 \frac{\lambda_{p1}}{2}; s_2 \frac{\lambda_{p2}}{2} \mid S \frac{\lambda'}{2} \rangle$ and $\langle L 0; S \frac{\lambda'}{2} \mid J \frac{\lambda'}{2} \rangle$ are Clebsh-Gordan coefficients of $SU(2)$ -group and the function $D_{\mu \lambda}^J(\varphi_k, \theta_k, -\varphi_k)$ is Wigner D -function with angle of vector $\hat{\mathbf{k}} = \mathbf{k} / |\mathbf{k}|$.

Using Wigner-Eckart theorem we have the Eq.(12) transform to

$$\begin{aligned} \langle \mathbf{k}', J', \mu' \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, J, \mu, \lambda_1, \lambda_2 \rangle &= \delta_{J,J'} \delta_{\mu,\mu'} V_{\lambda'_1, \lambda'_2; \lambda_1, \lambda_2}^J(\mathbf{k}', \mathbf{k}) , \\ V_{\lambda'_1, \lambda'_2; \lambda_1, \lambda_2}^J(\mathbf{k}', \mathbf{k}) &= \\ &= \int_{-1}^1 d(\cos \beta) \int_0^{2\pi} d\varphi D_{\lambda, \lambda'}^J(\varphi, \beta, -\varphi) \langle \mathbf{k}', \lambda'_1, \lambda'_2 \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 \rangle , \end{aligned} \quad (13)$$

where

$$\cos \beta = (\mathbf{k}\mathbf{k}') / (|\mathbf{k}| |\mathbf{k}'|) = \cos \theta_{k'} \cos \theta_k + \cos(\varphi_{k'} - \varphi_k) \sin \theta_{k'} \sin \theta_k . \quad (14)$$

The one of the tasks is to obtain kernel $V_{L,S;L',S'}^J(\mathbf{k}, \mathbf{k}')$ of radial equation (10) for relativistic fermion-fermion systems exactly in terms of scalar function.

Fermion-fermion QED potential

The interaction potential is constructed with the help of the scattering amplitude according to the below prescription *Lucha:1991*

1. Compute the scattering amplitude R_{fi} , which is defined in terms of the S -matrix element by the decomposition

$$S_{fi} = \delta_{fi} - i(2\pi) \delta(E_f - E_i) R_{fi}, \quad (15)$$

where i and f denote of initial and final state, respectively.

2. The potential \hat{V} can be extracted from Eq.(15) with the help of the relation

$$\langle f | \hat{V} | i \rangle = R_{fi}. \quad (16)$$

Therefore, by investigating the corresponding scattering problem of bound-state constituents, the potential (or part of potential) may be derived according to the above mentioned procedure. Let us illustrate the recipe of

potential calculation by applying to electron-proton bound system (Hydrogen atom).

We start with the process

$$e^-(k_1, \lambda_{k_1}) + p(k_2, \lambda_{k_2}) \rightarrow e^-(p_1, \lambda_{p_1}) + p(p_2, \lambda_{p_2}) . \quad (17)$$

where the momenta of the particles and spin numbers ($\lambda_{k_i} = \pm 1, \lambda_{p_i} = \pm 1$) are given between parentheses.

The initial approximation of the potential V for a point-like particles was selected in the form of the potential, which corresponding tree-level diagram, as depicted in Fig. 1

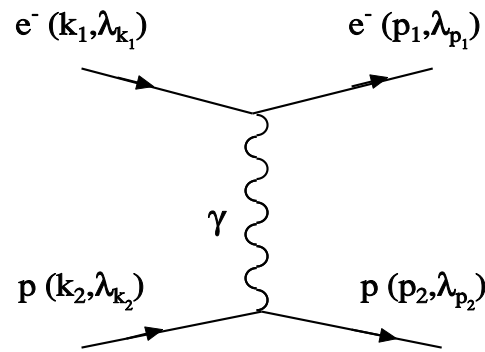


Figure 1: One-photon exchange of $e^- p$ elastic scattering

$$V_{1,\gamma} = \langle \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V} \parallel \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} \rangle = \frac{\alpha N_{k,k'}}{8\pi^2 q^2} \bar{u}_{\lambda_{p_1}}(p_1) \gamma_\mu u_{\lambda_{k_1}}(k_1) \bar{u}_{\lambda_{p_2}}(p_2) \gamma_\mu u_{\lambda_{k_2}}(k_2) . \quad (18)$$

The contribution of the Feynman diagrams graphed in Figs. 2-3 to potential matrix element of Eq.(8) in the centre-momentum system reads



Figure 2: Feynman diagrams of $e^- p$ elastic scattering

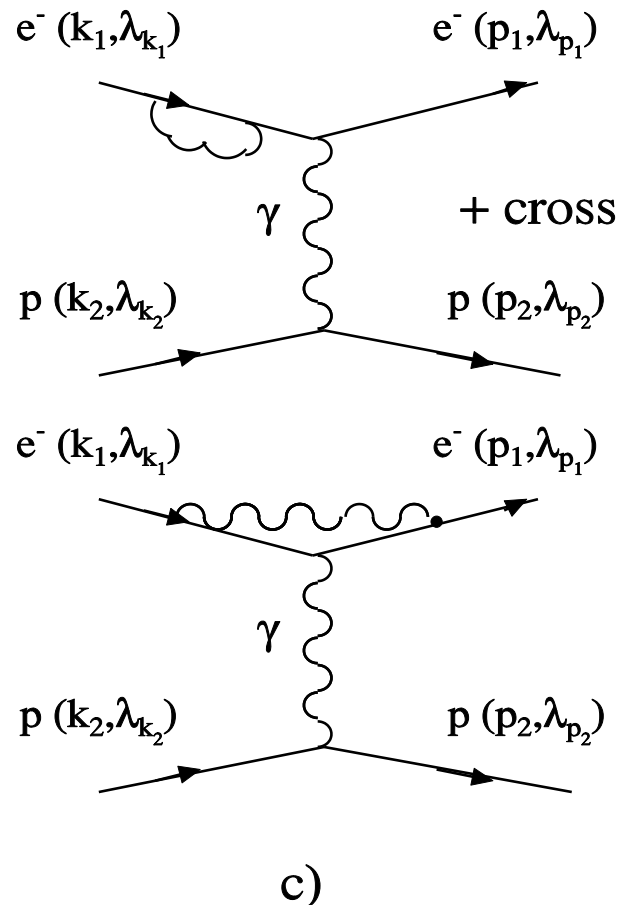


Figure 3: Feynman diagrams of contribution to the electron radius

$$\begin{aligned}
 \langle \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V} \parallel \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} \rangle = & \frac{\alpha N_{\mathbf{k}, \mathbf{k}'}}{8\pi^2 q^2 \left(1 + \hat{\Pi}^\gamma(\alpha, q^2)\right)} \\
 \bar{u}_{\lambda_{p_1}}(p_1) \left(F_1^e(q^2, \alpha) \gamma_\mu + \frac{F_2^e(q^2, \alpha)}{2m_2} i\sigma_{\mu\nu} q^\nu \right) u_{\lambda_{k_1}}(k_1) \times \\
 \times \bar{u}_{\lambda_{p_2}}(p_2) \left(F_1^p(q^2) \gamma_\mu - \frac{F_2^p(q^2)}{2m_2} i\sigma_{\mu\nu} q^\nu \right) u_{\lambda_{k_2}}(k_2) , \quad (19)
 \end{aligned}$$

where

$$\begin{aligned}
 k_1 = (\omega_{m_1}(\mathbf{k}), \mathbf{k}), \quad p_1 = (\omega_{m_1}(\mathbf{k}'), \mathbf{k}'), \\
 k_2 = (\omega_{m_2}(\mathbf{k}), -\mathbf{k}), \quad p_2 = (\omega_{m_2}(\mathbf{k}'), -\mathbf{k}') . \quad (20)
 \end{aligned}$$

$q = p_1 - k_1 = k_2 - p_2$ denotes the involved momentum transfer, α is fine structure constant after charge renormalization and

$$N_{\mathbf{k}, \mathbf{k}'} = 1 / \sqrt{\omega_{m_1}(\mathbf{k}) \omega_{m_1}(\mathbf{k}') \omega_{m_2}(\mathbf{k}) \omega_{m_2}(\mathbf{k}')}.$$

The square of momentum transfer

$$q^2 = (p_1 - k_1)^2 = (k_2 - p_2)^2 \quad (21)$$

which enters in the denominator of potential matrix element (13) transforms to

$$q^2 = -2k k' (z - \cos \beta) , \quad (22)$$

where

$$z = \frac{(\omega_{m_1}(\mathbf{k}) - \omega_{m_1}(\mathbf{k}')) (\omega_{m_2}(\mathbf{k}') - \omega_{m_2}(\mathbf{k})) + k^2 + k'^2}{2kk'} . \quad (23)$$

This treatment of q^2 allows for the correct Dirac limit in which the retardation contribution vanishes when one of the particles becomes infinitely heavy

Akhiezer:1965.

The photon-proton interaction is described by the Dirac ($F_1^p(q^2)$) and Pauli ($F_2^p(q^2)$) form factors and the photon-electron interaction graphed in Fig. 3 is described by the ($F_1^e(q^2, \alpha)$) and ($F_2^e(q^2, \alpha)$) form factors.

The spinor part of Eq.(20) is transformed as the scalar function with the help of the method of basis spinors (see, **Andreev:2004**).

We obtain that the relativistic potential (19) $V_{\lambda'_1, \lambda'_2; \lambda_1, \lambda_2}^J(k', k)$ for arbitrary J is the sum

$$V_{\lambda_{p_1}, \lambda_{p_2}; \lambda_{k_1}, \lambda_{k_2}}^J(k', k) = (-1) \frac{\alpha}{2\pi} \left[V_I^J(k', k) + \frac{1}{8m_1 m_2} V_{IV}^J(k', k) - \right. \\ \left. + \frac{(-1)}{12m_1} V_{II}^J(k', k) + \frac{1}{12m_2} V_{III}^J(k', k) \right], \quad (24)$$

where

$$\begin{aligned} V_I^J(\mathbf{k}', \mathbf{k}) = & \sum_{\sigma, \rho = -1}^1 \Upsilon_{-\rho \times \lambda_{k_2}}^{-\sigma \times \lambda_{k_1}}(\mathbf{k}) \Upsilon_{-\rho \times \lambda_{p_2}}^{-\sigma \times \lambda_{p_1}}(\mathbf{k}') \times \\ & \times \left[\delta_{\lambda_{k_1}, \lambda_{k_2}} G_{\lambda_{p_1}, -\lambda_{p_2}}^{-\lambda_{k_1}, \lambda_{k_1}} \left(J, \frac{1}{2}, \frac{1}{2}; \tilde{\Phi}(\mathbf{k}', \mathbf{k}) \right) + \right. \\ & \left. + \delta_{\sigma \times \lambda_{k_1}, \rho \times \lambda_{k_2}} G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}} \left(J, \frac{1}{2}, \frac{1}{2}; \tilde{\Phi}(\mathbf{k}', \mathbf{k}) \right) \right], \end{aligned} \quad (25)$$

with

$$\Upsilon_{\rho}^{\sigma}(\mathbf{k}) = \sqrt{\left(1 + \sigma \frac{k}{\omega_{m_1}(\mathbf{k})} \right) \left(1 + \rho \frac{k}{\omega_{m_2}(\mathbf{k})} \right)}. \quad (26)$$

$$V_{IV}^J(\mathbf{k}', \mathbf{k}) = \sum_{\sigma, \rho = -1}^1 \Upsilon_{-\rho \times \lambda_{k_2}}^{-\sigma \times \lambda_{k_1}}(\mathbf{k}) \Upsilon_{\rho \times \lambda_{p_2}}^{\sigma \times \lambda_{p_1}}(\mathbf{k}') \times$$
$$\times \left\{ \left[\mathbf{k}^2 + \mathbf{k}'^2 + (\omega_{m_1}(\mathbf{k}) + \omega_{m_1}(\mathbf{k}')) (\omega_{m_2}(\mathbf{k}) + \omega_{m_2}(\mathbf{k}')) \right] \times \right.$$
$$\left. G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}} \left(J, \frac{1}{2}, \frac{1}{2}; \tilde{\Phi}^{IV}(\mathbf{k}, \mathbf{k}') \right) + 2 \mathbf{k} \mathbf{k}' G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}} \left(J, \frac{1}{2}, \frac{1}{2}; Z^{IV}(\mathbf{k}, \mathbf{k}') \right) \right\},$$

(27)

$$\begin{aligned}
 V_{\text{II}}^J(\mathbf{k}, \mathbf{k}') &= \sum_{\sigma, \rho=-1}^1 \Upsilon_{-\rho \times \lambda_{k_2}}^{-\sigma \times \lambda_{k_1}}(\mathbf{k}) \Upsilon_{-\rho \times \lambda_{p_2}}^{\sigma \times \lambda_{p_1}}(\mathbf{k}') \left\{ (\lambda_{k_2} \ \rho \ \mathbf{k}') \times \right. \\
 &\times \left(3G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}}\left(J, \frac{1}{2}, \frac{1}{2}; Z^{\text{II}}(\mathbf{k}, \mathbf{k}')\right) - 2G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}}\left(J, \frac{1}{2}, \frac{3}{2}; \tilde{\Phi}^{\text{II}}(\mathbf{k}, \mathbf{k}')\right) \right) + \\
 &+ G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}}\left(J, \frac{1}{2}, \frac{1}{2}; \tilde{\Phi}^{\text{II}}(\mathbf{k}, \mathbf{k}')\right) \\
 &\left. [\rho (3k\lambda_{k_2} - 2k'\lambda_{p_2}) - 3(\omega_{m_1}(\mathbf{k}) + \omega_{m_1}(\mathbf{k}')))] \right\} . \tag{28}
 \end{aligned}$$

$$\begin{aligned}
 V_{\text{III}}^J(\mathbf{k}, \mathbf{k}') &= \sum_{\sigma, \rho=-1}^1 \Upsilon_{-\rho \times \lambda_{k_2}}^{-\sigma \times \lambda_{k_1}}(\mathbf{k}) \Upsilon_{\rho \times \lambda_{p_2}}^{-\sigma \times \lambda_{p_1}}(\mathbf{k}') \left\{ (\lambda_{k_1} \ \sigma \ \mathbf{k}') \times \right. \\
 &\times \left(3G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}}\left(J, \frac{1}{2}, \frac{1}{2}; Z^{\text{III}}(\mathbf{k}, \mathbf{k}')\right) - 2G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}}\left(J, \frac{1}{2}, \frac{3}{2}; \tilde{\Phi}^{\text{III}}(\mathbf{k}, \mathbf{k}')\right) \right) + \\
 &+ G_{\lambda_{p_1}, -\lambda_{p_2}}^{\lambda_{k_1}, -\lambda_{k_2}}\left(J, \frac{1}{2}, \frac{1}{2}; \tilde{\Phi}^{\text{III}}(\mathbf{k}, \mathbf{k}')\right) \\
 &\left. [\sigma (3k\lambda_{k_1} - 2k'\lambda_{p_1}) - 3(\omega_{m_2}(\mathbf{k}) + \omega_{m_2}(\mathbf{k}')))] \right\} . \tag{29}
 \end{aligned}$$

In eqs.(25)-(29) the function $G_{\lambda_{p_1}, \lambda_{p_2}}^{\lambda_{k_1}, \lambda_{k_2}} \left(J, s_1, s_2; \tilde{\Phi} (k', k) \right)$ is linear combination of $\tilde{\Phi}_L (k', k)$:

$$G_{\lambda_{p_1}, \lambda_{p_2}}^{\lambda_{k_1}, \lambda_{k_2}} \left(J, s_1, s_2; \tilde{\Phi} (k', k) \right) = \sum_{S=|s_1-s_2|}^{s_1+s_2} \sum_{L=|J-S|}^{J+S} \frac{(2L+1)}{(2J+1)} \left\langle s_1 \frac{\lambda_{k_1}}{2}; s_2 \frac{\lambda_{k_2}}{2} \middle| S \frac{\lambda}{2} \right\rangle \left\langle s_1 \frac{\lambda_{p_1}}{2}; s_2 \frac{\lambda_{p_2}}{2} \middle| S \frac{\lambda'}{2} \right\rangle \left\langle L 0; S \frac{\lambda'}{2} \middle| J \frac{\lambda'}{2} \right\rangle \left\langle L 0; S \frac{\lambda}{2} \middle| J \frac{\lambda}{2} \right\rangle \tilde{\Phi}_L (k', k) . \quad (30)$$

We use the following notation

$$Z_L (k', k) = \frac{1}{2L+1} \left((L+1) \tilde{\Phi}_{L+1} (k', k) + L \tilde{\Phi}_{L-1} (k', k) \right) , \quad (31)$$

and

$$\begin{aligned} \tilde{\Phi}_L^I(k', k) = & \int_{-1}^1 \frac{P_L(x)}{q^2 \left(1 + \hat{\Pi}^\gamma(\alpha, q^2)\right)} \\ & (F_1^p(q^2) - F_2^p(q^2)) (F_1^e(q^2, \alpha) + F_2^e(q^2, \alpha)) dx, \end{aligned} \quad (32)$$

$$\tilde{\Phi}_L^{II}(k', k) = \int_{-1}^1 \frac{P_L(x) F_2^p(q^2) (F_1^e(q^2, \alpha) + F_2^e(q^2, \alpha))}{q^2 \left(1 + \hat{\Pi}^\gamma(\alpha, q^2)\right)} dx, \quad (33)$$

$$\tilde{\Phi}_L^{III}(k', k) = \int_{-1}^1 \frac{P_L(x) (F_1^p(q^2) - F_2^p(q^2)) F_2^e(q^2, \alpha)}{q^2 \left(1 + \hat{\Pi}^\gamma(\alpha, q^2)\right)} dx, \quad (34)$$

$$\tilde{\Phi}_L^{IV}(\mathbf{k}', \mathbf{k}) = \int_{-1}^1 F_2^p(q^2) F_2^e(q^2, \alpha) \frac{P_L(x)}{q^2} dx, \quad (35)$$

The relations (24)-(35) determines the potential of two fermion system with arbitrary total angular momentum J , orbital momentum L and spin moment $S = 0, 1$.

Method of numerical calculations

There are several methods of calculating bound states. The standard way is to directly solve the differential equation in configuration space. Unfortunately we cannot use this method in the relativistic case, since we have to deal with a term that contains a square root of the differential operator $T(\mathbf{k})$. Another intractable problem is to obtain the configuration-space interaction potential exactly, as the Fourier transformation of the momentum-space potential $V^J(k', k)$.

Another possible method is to directly discretize the integral equation in momentum space (see, for example *Deloff:2006*, *Jean:1994*, *Tang:2001*, *Skachkov:1981*). In this article we use method in which all integral equations are solved numerically using semi-spectral Chebyshev method *Deloff:2006*.

In the case when $k, k' \ll m_1^2, m_2^2$ potential (24) transform to ordinary

Coulomb potential in momentum space:

$$V^J(k, k') = -\frac{\alpha}{\pi} \frac{Q_J(y)}{k k'} , \quad (36)$$

where $Q_L(y)$ is the Legendre polynomial of second kind and

$$y = \frac{k^2 + k'^2}{2k k'} . \quad (37)$$

We have done numerical calculations of nonrelativistic Schrödinger equation in momentum space

$$\frac{\alpha}{\pi} \int_0^\infty \frac{Q_J(y)}{k k'} \Phi^J(k) dk = \left(\epsilon - \frac{k'^2}{2m_{red}} \right) \Phi^J(k') , \quad (38)$$

with the reduced mass m_{red} .

The result is given in table 1.

Table 1: Absolute differences between Eigen-energies in eV of the Hydrogen atom according to the nonrelativistic Schrödinger equation with $J = 0$ and those of calculated numerically for different numbers of mesh points N in momentum space.

n	Calculated			Exact
	$N=20$	$N=40$	$N=80$	$-\frac{\alpha^2 m_1 m_2}{2n^2(m_1+m_2)}$
1	$1.8 * 10^{-8}$	$5.5 * 10^{-11}$	$1.0 * 10^{-13}$	-13.598286461157846
2	$6.2 * 10^{-7}$	$8.8 * 10^{-10}$	$3.4 * 10^{-12}$	-3.3995716152894615
3	$9.3 * 10^{-6}$	$6.6 * 10^{-9}$	$2.6 * 10^{-11}$	-1.5109207179064272
4	$2.4 * 10^{-4}$	$2.5 * 10^{-8}$	$1.1 * 10^{-10}$	-0.8498929038223654
5	$5.9 * 10^{-3}$	$1.1 * 10^{-7}$	$3.4 * 10^{-10}$	-0.5439314584463139

As we can see from Table 1, differences between Eigen-energies of the exact

solutions for Hydrogen atom according to the nonrelativistic Schrödinger equation in momentum space with $J = 0$ and those of calculated numerically at $N = 80$ are about 25 Hz for $n = 1$ and 100 kHz for $n = 5$. This precision is of the high order of magnitude as it's for experimental precision . **Therefore, numerical solution of the bound state equations in momentum space give the possibility to analyse different corrections.**

One-loop corrections

The contribution of the one-loop vacuum polarization (Feynman diagrams graphed in Fig. 2.b) to potential matrix element of Eq.(18) reads

$$\Delta V_{loop-1} = \langle \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \Delta \hat{V}_{loop} \parallel \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} \rangle = \frac{(-1)\alpha N_{\mathbf{k},\mathbf{k}'} \hat{\Pi}^\gamma (q^2)}{8\pi^2 q^2} \bar{u}_{\lambda_{p_1}} (p_1) \gamma_\mu u_{\lambda_{k_1}} (k_1) \bar{u}_{\lambda_{p_2}} (p_2) \gamma_\mu u_{\lambda_{k_2}} (k_2) , \quad (39)$$

The function $\hat{\Pi}^\gamma (q^2)$ is renormalization electron vacuum polarization and we have that

$$\hat{\Pi}^\gamma (q^2) = \frac{\alpha}{3\pi} \left[\left(3 - \frac{1}{b^2} \right) \left(1 - \frac{\text{arcth}(b)}{b} \right) - \frac{1}{3} \right] \quad (40)$$

with

$$b = \frac{|q|}{\sqrt{|q|^2 + 4m_e^2}} . \quad (41)$$

If

$$(-q^2) \sim m_e^2 (Z\alpha)^2 \quad (42)$$

(electronic hydrogen) and $q^2 \ll m_e^2$, then, we obtain ***Akhiezer:1965, Salpeter:1957***

$$\hat{\Pi}^\gamma(q^2) = \frac{\alpha}{3\pi} \frac{q^2}{5m_e^2}. \quad (43)$$

However, if

$$q^2 \sim m_\mu^2 (Z\alpha)^2$$

(muonic hydrogen, m_μ is the mass of the muon), then we cannot use (43) (see, ***Martynenko:2004***). With muonic hydrogen, we must construct the particle interaction operator in the one-photon approximation using exact expression (40).

With the help of the eqs. (32)-(35) we obtain that the function $\tilde{\Phi}_L(q^2)$ for

potential (39) can be written as follows:

$$\begin{aligned}\tilde{\Phi}_\ell^{loop}(k', k) &= \int_{-1}^1 dx \frac{\hat{\Pi}^\gamma(q^2(z, x)) P_\ell(x)}{q^2(z, x)} = \\ &= F_\ell(z, 1) - F_\ell(z, -1) ,\end{aligned}\quad (44)$$

where $P_\ell(y)$ is the Legendre polynomial.

The function (32) for electron loop corrections we represent in the form

As example, for $\ell = 0$ we have that

$$\begin{aligned}F_{\ell=0}(z, y) &= \frac{1}{3k k' (z - y) b} \left[2 m_e^2 + 5k k' (z - y) \operatorname{arctanh}(b) + \right. \\ &\left. + (2 m_e^2 + 3k k' (z - y) \operatorname{arctanh}^2(b)) b \right] ,\end{aligned}\quad (45)$$

$$b = \frac{\sqrt{k k' (z - y)}}{\sqrt{2 m_e^2 + k k' (z - y)}} .\quad (46)$$

We now consider a simple case of two fermion system with $J = L = S = 0$.

As result we obtain equation for Hydrogen-like system with kernel $V_{1,\gamma}^{J=0}(k, k')$, which is connect with potential matrix element (18) and equation with kernel $V_{loop-1}^{J=0}(k, k')$, which is connect with the sum of potential matrix elements (18) and (39).

In relativistic case the exact contribution of the electron vacuum polarization can be found with the use of equation

$$\Delta E_{loop-1} = \tilde{E} - E, \quad (47)$$

where E are eigenvalues of Eq. (10) with kernel (18) and \tilde{E} are eigenvalues of Eq. (10) with kernel (18) plus kernel (39) with vacuum polarization $\hat{\Pi}^\gamma(q^2)$ (40).

For nonrelativistic case the one-loop contribution with $\hat{\Pi}^\gamma(q^2)$ (43) can be

found as *Akhiezer:1965, Salpeter:1957, Eides:2000*

$$\Delta E_{\text{nonrel}}^{\text{loop-1}} = -\frac{4\alpha (Z\alpha)^4}{15\pi n^3 m_e^2} m_{\text{red}}^3 \delta_{L0} , \quad (48)$$

where

$$m_{\text{red}} = (m_1 m_2) / m_{12} , m_{12} = m_1 + m_2 . \quad (49)$$

As follows from the Table 2 corrections $\Delta E_{\text{relcor.}} = \Delta E_{\text{nonrel}}^{\text{loop-1}} - \Delta E_{\text{loop-1}}$ which are connected with the exact equation for electron one-loop vacuum polarization (40) are $\sim 0.51 - 0.56\%$ for all states and therefore these contributions are the same order of magnitude as it's for experimental precision.

Table 2: Electronic one-loop corrections in MHz of the nonrelativistic equation (48) and of the exact relativistic equation (40) for Hydrogen atom ($e - p$ system).

n	Nonrel., $\Delta E_{\text{nonrel}}^{\text{loop}-1}$	Exact, $\Delta E_{\text{rel}}^{\text{loop}-1}$	Rel. corr., $\Delta E_{\text{relcor.}}$
1	-216.676	-215.513	1.16264
2	-27.0845	-26.9334	0.151076
3	-8.02503	-7.9834	0.041636

As can see from Table 3, the electronic vacuum polarization effects are very sensitive to the bound state structure because the characteristic momentum of the particles in muonic hydrogen atom is $\sim (200 m_e Z \alpha)$. So, the contribution of the vacuum polarization amplitudes in one-photon interaction cannot be expressed by a simple factor $|\psi(0)|^2 \sim 1/n^3$ (48) but it is dependent on principal quantum number n in more complicated form. Our

result for muonic hydrogen atom is agreement with the ones of Martynenko *Martynenko:2004*.

Table 3: Electronic one-loop corrections in eV of the nonrelativistic equation (48) and of the exact relativistic equation (40) for muonic hydrogen (μp).

n	Nonrel., $\Delta E_{\text{nonrel}}^{\text{loop}-1}$	Exact, $\Delta E_{\text{rel}}^{\text{loop}-1}$
1	-5.76088	-1.908150
2	-0.72011	-0.220736
3	-0.21337	-0.064608

Electron electromagnetic structure corrections

The next contribution of order α^2 to the potential is connected with electromagnetic structure of electron (muon)(see Fig. 3).

The contribution of the Feynman diagrams graphed in Fig3 to potential matrix element of Eq.(8) in the centre-momentum system reads

$$\begin{aligned} \langle \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \| \hat{V} \| \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} \rangle = & \frac{\alpha N_{\mathbf{k}, \mathbf{k}'}}{8\pi^2 q^2} \\ & \bar{u}_{\lambda_{p_1}}(p_1) \left(F_1^e(q^2, \alpha) \gamma_\mu + \frac{F_2^e(q^2, \alpha)}{2m_2} i\sigma_{\mu\nu} q^\nu \right) u_{\lambda_{k_1}}(k_1) \times \\ & \times \bar{u}_{\lambda_{p_2}}(p_2) \gamma_\mu u_{\lambda_{k_2}}(k_2) , \end{aligned} \quad (50)$$

where

$$F_1^e(q^2) = \frac{\alpha}{\pi} \left[\left(\ln \left(\frac{m_1}{\lambda} \right) - 1 \right) + \frac{\operatorname{arcth}(b)}{4b} \left[(1 + b^2) \ln \left(\frac{1 - b^2}{4} \right) + \right. \right. \\ \left. \left. + 2 \left(2 + b^2 - 2(1 + b^2) \ln \left(\frac{m_1}{\lambda} \right) \right) \right] + \right. \\ \left. \frac{(1 + b^2)}{4b} \left(\operatorname{Li}_2 \left(\frac{1 + b}{2} \right) - \operatorname{Li}_2 \left(\frac{1 - b}{2} \right) \right) \right], \quad (51)$$

$$F_2^e(q^2) = \frac{\alpha}{2\pi} \frac{(1 - b^2) \operatorname{arcth}(b)}{b} \quad (52)$$

with the mass of photon λ .

Numerical calculation of corrections for this case is presented in table 4

Table 4: Electronic structure corrections in MHz of the relativistic equation (50) for Hydrogen atom ($e - p$ system).

n	ΔE for F_2^e	Total ΔE for F_1^e and F_2^e
1	0.397695	8.266304
2	0.049744	1.056550
3	0.014713	0.314855

Summary

Semi-spectral Chebyshev method of numerical calculation can be used for solving of integral equations of relativistic bound state with high precision (25 – 60 Hz for $n = 1$ and 100 kHz for $n = 5$).

As example, the leading order contributions to the electron radius and polarization insertion was calculated using relativistic expressions for Hydrogen ($e - p$).