The Bethe-Salpeter equation with bosons and fermions.

G.V.Efimov

Bogoliubov Laboratory of Theoretical Physics

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1. The Bethe-Salpeter (BS) equation in the ladder approximation is studied within a boson and fermion theory.

2. The BS kernel in a symmetric form looks as

$$K = \begin{cases} K_I & \text{bosons} \\ \\ K_0 + K_I & \text{fermions} \end{cases}$$

$\mathrm{Tr}\ K_0^2 = \infty$ is of the "fall at center" potential type.

$\mathrm{Tr}\ K_I^2 < \infty$ is responsible for bound states

3. Variational procedure of calculations is formulated.

4. The binding energy of the 1^+-state (deutron) in the quantum scalar and pseudoscalar mesodynamics is calculated.

$$\textbf{Interaction}: \begin{cases} L_I = g\Phi^+\Phi\phi & \text{bosons} \\ \\ L_I = g(\overline{\Psi}\Gamma\Psi)\phi & \text{fermions} \end{cases}$$

$$\begin{aligned} & A(x_1, x_2; x_3, x_4) \\ & = g^2\Gamma D(x_1 - x_2)\Gamma\delta(x_1 - x_3)\delta(x_2 - x_4) \\ & + g^2 D(x_1 - x_2)\iint dz_1 dz_2 \Gamma S(x_1 - z_1)\Gamma S(x_2 - z_2) \\ & \cdot A(z_1, z_2; x_3, x_4). \end{aligned}$$

$$S_{\Psi\Psi\to\Psi\Psi} = \int ... \int dx_1 ... dx_4 \overline{\Psi}(x_1)\overline{\Psi}(x_2)A(x_1, x_2; x_3, x_4)\Psi(x_3)\Psi(x_4),$$

$$\overline{\Psi}_\alpha\overline{\Psi}_\beta = \sum_J (\overline{\Psi}\Gamma^J\Psi^C)(C\Gamma^J)_{\beta\alpha}, \quad \Psi_\alpha\Psi_\beta = \sum_J (\overline{\Psi^C}\Gamma^J\Psi)(\Gamma^J C^\top)_{\beta\alpha}$$

$$\Gamma^S = I,\ \Gamma^V = \gamma_\mu,\ \Gamma^T = \sigma_{\mu\nu},\ \Gamma^A = \gamma_5\gamma_\mu,\ \Gamma^P = i\gamma_5$$

$$S_{\Psi\Psi\to\Psi\Psi}$$

$$= \sum_{J_1 J_2} (\overline{\Psi}(x_1)\Gamma^{J_1}\Psi^C(x_2)) A^{J_1 J_2}(x_1, x_2; x_3, x_4)(\overline{\Psi^C}(x_3)\Gamma^{J_2}\Psi(x_4))$$

$$A^{J_1 J_2}(x_1, x_2; x_3, x_4) = \mathrm{Tr}(C\Gamma^J)A(x_1, x_2; x_3, x_4)(\Gamma^J C^\top)$$

$$x_1 = x + \frac{y}{2}, \quad x_3 = x' + \frac{y'}{2}, \quad z_1 = x'' + \frac{y''}{2}$$

$$x_2 = x - \frac{y}{2}, \quad x_4 = x' - \frac{y'}{2}, \quad z_2 = x'' - \frac{y''}{2}$$

$$\tilde{A}^{JJ'}(p; y, y') = \delta^{JJ'} g^2 D(y)\delta(y - y')$$

$$+g^2 D(y) \int dy'' \sum_{J_1} \Pi^{JJ_1}(p; y - y'')\tilde{A}^{J_1 J'}(p; y'', y')$$

$$\Pi^{JJ'}(p, y) \sim \int dx \; e^{ipx} \mathrm{Tr}\left[S\left(x + \frac{y}{2}\right)\Gamma^J S\left(-x + \frac{y}{2}\right)\Gamma^{J'}\right].$$

$$\tilde{A}^{JJ'}(p;y,y') = g^2\sqrt{D(y)}\mathcal{A}^{JJ'}(p;y,y')\sqrt{D(y')}$$

$$\mathcal{A}^{JJ'}(p;y,y') = \delta^{JJ'}\delta(y-y') + g^2 \sum\!\!\!\!\!\!\int_{J_1,y''} \mathcal{K}^{JJ_1}(p;y,y'')\mathcal{A}^{J_1J'}(p;y'',y'),$$

$$\mathcal{K}^{JJ'}(p;y,y') = \sqrt{D(y)}\tilde{\Pi}^{JJ'}(p;y-y')\sqrt{D(y')}$$

$\mathcal{K}^{JJ'}(p;y,y')$ is Hermitian for $p^2 < (2m)^2$.

$$\{U_Q(p,y)\}: \quad \Lambda_Q(p^2)U_Q^J(p;y) = \sum_{J'}\int dy'\mathcal{K}^{JJ'}(p;y,y')U_Q^{J'}(p;y')$$

$$\tilde{A}^{JJ'}(p;y,y') = \sum_Q \sqrt{D(y)}U_Q^J(p;y)\frac{g^2}{I - g^2\Lambda_Q(p^2)}U_Q^{J'}(p;y')\sqrt{D(y')}$$

Diagonalization of the kernel

$$\Lambda_Q(p^2)\mathcal{Y}_{\mathcal{Q}}{}^{J}(y) = \sum_{JJ'} \int dy' \; \mathcal{K}^{JJ'}(y,y')\mathcal{Y}_Q^{J'}(y'), \qquad \Lambda_Q(M^2) = 1.$$

$$\mathcal{K}(y,y') \sim \sqrt{D(y)} \int \frac{dk}{(2\pi)^4} \frac{e^{-ik(y-y')}[k^2 C_1 + m^2 C_2]}{\left(k^2 + m^2 - \frac{M^2}{4}\right)^2 + M^2 k_4^2} \sqrt{D(y')}$$

$$\begin{cases} C_1 = 0 \quad \Rightarrow \quad \mathcal{K}(y,y') \sim \ln(y-y')^2 \quad \text{bosons}, \\ C_2 \neq 0 \quad \Rightarrow \quad \mathcal{K}(y,y') \sim \frac{1}{(y-y')^2} \qquad \text{fermions}, \end{cases}$$

$$\begin{cases} Boson \qquad \mathrm{Tr}\; \mathcal{K}^2 = \iint dy dy' \; \mathcal{K}(y,y')\mathcal{K}(y',y) < \infty, \\ Fermion \quad \mathrm{Tr}\; \mathcal{K}^2 = \iint dy dy' \; \mathcal{K}(y,y')\mathcal{K}(y',y) = \infty \end{cases}$$

Fermions

$$\Lambda \cdot U(y) = g^2 \int dy' \ \mathcal{K}(y,y') \cdot U(y'), \qquad \Lambda = 1$$

$$\mathcal{K}(y,y') = \mathcal{K}_0(y,y') + \mathcal{K}_I(y,y')$$

$$\mathcal{K}_0(y,y') = \sqrt{D(y)} \int \frac{dk}{(2\pi)^4} \frac{e^{-ik(y-y')} k^2 C_1}{\left(k^2 + m^2 - \frac{M^2}{4}\right)^2 + M^2 k_4^2} \sqrt{D(y')},$$

$$\mathcal{K}_I(y,y') = \sqrt{D(y)} \int \frac{dk}{(2\pi)^4} \frac{e^{-ik(y-y')} m^2 C_2}{\left(k^2 + m^2 - \frac{M^2}{4}\right)^2 + M^2 k_4^2} \sqrt{D(y')}$$

$$\mathrm{Tr}\ \mathcal{K}_0^2 = \iint dy dy' \ \mathcal{K}_0(y,y')\mathcal{K}_0(y',y) = \infty,$$

$$\mathrm{Tr}\ \mathcal{K}_I^2 = \iint dy dy' \ \mathcal{K}_I(y,y')\mathcal{K}_I(y',y) < \infty.$$

$$\Lambda(g^2)U = g^2 \left[\mathcal{K}_0 + \mathcal{K}_I\right] U.$$

The kernel $\mathcal{K}_0$

$$U(y) = g^2 \int dy' \; \mathcal{K}_0(y,y') \cdot U(y'), \qquad (\Lambda(g^2) = 1)$$

$$\mathcal{K}_0(y,y') = \sqrt{D(y)} \left[\frac{C_1}{(2\pi)^2 (y-y')^2} + R(y-y') \right] \sqrt{D(y')} \Psi(y')$$

$$U(y) = \sqrt{D(y)} \Phi(y)$$

$$\left[-\Box_y - \frac{g^2 C_1}{y^2} \right] \Phi(y) = p^2 \Phi(y)$$

$$\Phi(r) \sim \frac{1}{r} J_\nu(pr), \quad \nu = \sqrt{1 - g^2 C_1}, \quad g^2 < g_c^2 = \frac{1}{C_1}$$

The kernel $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_I$

$\mathcal{K}_0$ - continuous spectrum
$\mathcal{K}_I$ - bound states

$$U = g^2[\mathcal{K}_0 + \mathcal{K}_I] \cdot U \quad \Rightarrow \quad [I - g^2\mathcal{K}_0]U = g^2\mathcal{K}_I \cdot U$$

$$U = \frac{1}{\sqrt{I - g^2\mathcal{K}_0}} A, \qquad A = g^2\mathcal{K}_G \cdot A$$

$$\mathcal{K}_G = \frac{1}{\sqrt{I - g^2\mathcal{K}_0}} \mathcal{K}_I \frac{1}{\sqrt{I - g^2\mathcal{K}_0}}, \quad \mathrm{Tr}\ \mathcal{K}_G^2 < 0$$

$$g^2 < g_c^2$$

Variational calculations

$$
\begin{aligned}
1 &= g^2 \max_A \frac{\left(A \frac{1}{\sqrt{I-g^2\mathcal{K}_0}} \mathcal{K}_I \frac{1}{\sqrt{I-g^2\mathcal{K}_0}} A\right)}{(AA)} \\
&= g^2 \max_U \frac{(U\mathcal{K}_I U)}{(U\left[I-g^2\mathcal{K}_0\right]U)} \\
&= g^2 \max_\Phi \frac{(\Phi D\mathcal{R}_I D\Phi)}{(\Phi\left[D-g^2 D\mathcal{R}_0 D\right]\Phi)} \\
&= g^2 \max_V \frac{(V\mathcal{R}_I V)}{\left(V\left[\frac{1}{D}-g^2\mathcal{R}_0\right]V\right)}.
\end{aligned}
$$

Mesodynamics

$$\mathcal{L}(x) = (\overline{N}(\hat{p} - m)N) - \frac{1}{2}(\boldsymbol{\pi}(-\partial^2 + \mu^2)\boldsymbol{\pi}) - \frac{1}{2}(\phi(-\partial^2 + \mu^2)\phi)$$

$$-g_P(\overline{N}i\gamma_5\boldsymbol{\tau}N)\boldsymbol{\pi} - g_S(\overline{N}N)\phi$$

$$\begin{cases} (\overline{N^C}(x)\Gamma^J N(x)) \neq 0 & \text{for} \quad T = 0, \quad J = V,\ T, \\ (\overline{N^C}(x)\Gamma^J \boldsymbol{\tau} N(x)) \neq 0 & \text{for} \quad T = 1, \quad J = S,\ A,\ P. \end{cases}$$

Nucleon-nucleon states 1^+

	$(\overline{N^C}\Gamma^J N), \quad T=0$	L	S	J	J^P
V	$\left(\overline{N^C}\gamma_\mu N\right)$				
	$\left(\overline{N^C}\boldsymbol{\gamma} N\right) \Longrightarrow \left(\eta_1^+\boldsymbol{\sigma}\chi_2\right)\left(w_1^+ v_2\right)$	0	1	1	1^+
	$\left(\overline{N^C}\gamma_0 N\right) \Longrightarrow \left(\eta_1^+(\boldsymbol{\sigma}\boldsymbol{k})\chi_2\right)\left(w_1^+ v_2\right)$	1	1	0	0^-
T	$\left(\overline{N^C}\sigma_{\mu\nu} N\right)$				
	$\left(\overline{N^C}\sigma_{ij} N\right) \Longrightarrow \left(\eta_1^+[\boldsymbol{\sigma}\times\boldsymbol{k}]\chi_2\right)\left(w_1^+ v_2\right)$	1	1	1	1^-
	$\left(\overline{N^C}\sigma_{0j} N\right) \Longrightarrow \left(\eta_1^+\boldsymbol{\sigma}\chi_2\right)\left(w_1^+ v_2\right)$	0	1	1	1^+

$$\mathbf{D} \propto \cos\theta \left(\overline{N^C}\boldsymbol{\gamma} N\right) + \sin\theta \left(\overline{N^C} i\gamma_0\boldsymbol{\gamma} N\right)$$

The scalar interaction

$$\begin{pmatrix} A^{VV} & A^{VT} \\ A^{TV} & A^{TT} \end{pmatrix} = I_S + g_S^2 \begin{pmatrix} \Pi^{VV} & \Pi^{VT} \\ \Pi^{TV} & \Pi^{TT} \end{pmatrix} \begin{pmatrix} A^{VV} & A^{VT} \\ A^{TV} & A^{TT} \end{pmatrix}. \tag{1}$$

The pseudoscalar interaction

$$\begin{pmatrix} A^{VV} & A^{VT} \\ A^{TV} & A^{TT} \end{pmatrix} = I_P + g_P^2 \begin{pmatrix} \Pi^{VV} & \Pi^{VT} \\ -\Pi^{TV} & -\Pi^{TT} \end{pmatrix} \begin{pmatrix} A^{VV} & A^{VT} \\ A^{TV} & A^{TT} \end{pmatrix}.$$

$$A = I + TA$$

$$T^{(S)} = \begin{pmatrix} m^2 + \frac{M^2}{4} & imM \\ -imM & m^2 + \frac{M^2}{4} \end{pmatrix} \;\Rightarrow\; \Lambda_\pm = \left(m \pm \frac{M}{2}\right)^2$$

$$T^{(P)} = \begin{pmatrix} m^2 + \frac{M^2}{4} & -mM \\ mM & -\left(m^2 + \frac{M^2}{4}\right) \end{pmatrix} \;\Rightarrow\; \Lambda_\pm = \pm\left(m - \frac{M}{2}\right)^2 .$$

Bound state: $\Lambda = \left(m + \frac{M}{2}\right)^2$.

Test function V and numerical calculations

$$\text{Vertex} \quad V(\mathbf{y}, y_4) = \sqrt{D\left(\mathbf{y}^2 + y_4^2\right)} \cdot U(\mathbf{y}, y_4)$$

$$\begin{cases} 1 = 3g^2 \max\limits_V \frac{(V\Pi_I V)}{\left(V \frac{1}{D} V\right) - 3g^2 (V\Pi_0 V)}, \\ \\ 0 = (V\Pi_{AB} V). \end{cases}$$

$$\begin{aligned} \tilde{V}(k) &= \frac{1}{a^2(\mathbf{k}^2 + b^2 k_4^2) + \mu^2}, \\ V(x) &= \frac{1}{a^4 b} \cdot \frac{\mu^2}{(2\pi)^2} \frac{K_1(z(x))}{z(x)}, \qquad z(x) = \frac{\mu}{a}\sqrt{\mathbf{x}^2 + \frac{x_4^2}{b^2}}. \end{aligned}$$

$$\Psi(\mathbf{x}, x_4) \sim \frac{V(\mathbf{x}, x_4)}{D(x)} \sim e^{-\mu\left(\frac{1}{a} - 1\right)|\mathbf{x}| - \mu\left(\frac{1}{ab} - 1\right)|x_4|}$$

Binding energy of the state 1^+ (deutron) as a function
of the coupling constant $\alpha_g = \frac{g^2}{4\pi^2}$ for $\mu = 139\ Mev$, $m = 938\ Mev$.

$\varepsilon = M - 2m$ (Mev)	$\alpha_g = \frac{g^2}{4\pi}$	a	b	$\cos 2\theta$	$\delta = \frac{\alpha_g}{2\pi} \cdot \frac{\mathcal{W}_0}{\mathcal{W}}$
0.001	0.093	0.692	1.127	4.7·10^{-4}	2.6·10^{-7}
0.01	0.093	0.690	1.128	4.7·10^{-4}	2.7·10^{-7}
0.10	0.097	0.677	1.133	5.1·10^{-4}	3.1·10^{-7}
0.50	0.109	0.639	1.149	6.4·10^{-4}	4.9·10^{-7}
1.00	0.119	0.613	1.159	7.6·10^{-4}	6.8·10^{-7}
2.23	**0.135**	**0.576**	**1.172**	**9.5·10^{-4}**	**1.1·10^{-6}**
5.00	0.160	0.530	1.186	1.3·10^{-3}	2.0·10^{-6}
10.00	0.192	0.485	1.196	1.8·10^{-3}	3.9·10^{-6}
25.00	0.265	0.415	1.200	3.1·10^{-3}	1.2·10^{-6}
50.00	0.358	0.358	1.191	5.3·10^{-3}	3.3·10^{-5}
100.00	0.516	0.300	1.168	9.6·10^{-3}	1.0·10^{-4}

Binding energy of the state 1^+ (deutron) as a function of the coupling constant $\alpha_g = \frac{g^2}{4\pi^2}$ for $\mu = 600\ Mev$, $m = 938\ Mev$.

$\varepsilon = M - 2m$ (Mev)	$\alpha_g = \frac{g^2}{4\pi}$	a	b	$\cos 2\theta$	$\delta = \frac{\alpha_g}{2\pi} \cdot \frac{\mathcal{W}_0}{\mathcal{W}}$
0.10	0.469	0.760	1.060	$9.8{\cdot}10^{-3}$	$1.0{\cdot}10^{-4}$
0.50	0.487	0.750	1.062	$1.0{\cdot}10^{-2}$	$1.2{\cdot}10^{-4}$
1.00	0.502	0.742	1.064	$1.1{\cdot}10^{-2}$	$1.4{\cdot}10^{-4}$
2.23	**0.525**	**0.731**	**1.066**	$\mathbf{1.2{\cdot}10^{-2}}$	$\mathbf{1.6{\cdot}10^{-4}}$
5.00	0.562	0.714	1.069	$1.3{\cdot}10^{-2}$	$2.0{\cdot}10^{-4}$
10.00	0.610	0.695	1.071	$1.5{\cdot}10^{-2}$	$2.5{\cdot}10^{-4}$
50.00	0.842	0.627	1.074	$2.4{\cdot}10^{-2}$	$6.7{\cdot}10^{-4}$
100.00	1.051	0.586	1.070	$3.3{\cdot}10^{-2}$	$1.3{\cdot}10^{-3}$

Nonrelativistic approach

$$\Phi(y) = \Phi(\mathbf{y}, y_4) \implies \Psi(\mathbf{y})$$

$$(\Phi D \mathcal{R}_I D \Phi) \implies$$

$$\iint d\mathbf{y} d\mathbf{y}' \frac{\Psi(\mathbf{y})}{|\mathbf{y}|} \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{dk_4}{2\pi} \frac{e^{-(|\mathbf{y}|+|\mathbf{y}'|)\sqrt{\mu^2+k_4^2}+i\mathbf{k}(\mathbf{y}-\mathbf{y}')}}{(\mathbf{k}^2+k_4^2+\Delta)^2+M^2k_4^2} \frac{\Psi(\mathbf{y}')}{|\mathbf{y}'|}$$

$$\langle \mathbf{k}^2 \rangle \sim \mu^2, \quad \langle k_4^2 \rangle \sim \frac{1}{M^2} \big(\langle \mathbf{k}^2 \rangle + \langle k_4^2 \rangle + \Delta \big)^2 \sim \mu^2 \cdot \frac{\mu^2}{M^2} \ll \mu^2$$

$$(I) \quad \frac{\epsilon}{m} \ll 1, \qquad\qquad (II) \quad \frac{\mu}{m} \ll 1$$

Yukawa potential approach

$$H = \frac{p^2}{m} - \alpha \frac{e^{-\mu r}}{r}, \qquad \alpha = \frac{3}{\sqrt{2}}\alpha_g$$

$$\left[\frac{p^2}{m} - \alpha \frac{e^{-\mu r}}{r}\right]\Psi(r) = -\varepsilon\Psi(r), \qquad \left[\frac{p^2}{m} + \varepsilon\right]\Psi(r) = \alpha \frac{e^{-\mu r}}{r}\Psi(r),$$

$$\Psi(r) = \frac{\alpha}{\frac{p^2}{m} + \varepsilon}\frac{e^{-\mu r}}{r}\Psi(r), \qquad \Psi(r) = \sqrt{\frac{e^{-\mu r}}{r}}\Phi(r),$$

$$\Phi(r) = \alpha\sqrt{\frac{e^{-\mu r}}{r}}\frac{1}{\frac{p^2}{m} + \varepsilon}\sqrt{\frac{e^{-\mu r}}{r}}\Phi(r)$$

$$1 = \alpha \max_{\Phi} \frac{\left(\Phi\sqrt{\frac{e^{-\mu r}}{r}}\frac{1}{\frac{p^2}{m}+\varepsilon}\sqrt{\frac{e^{-\mu r}}{r}}\Phi\right)}{(\Phi\Phi)}$$

$$m = 938\ Mev, \qquad \epsilon = 2.23\ Mev$$

$$\frac{\epsilon}{m} = 2.4 \cdot 10^{-3}$$

$\mu\ (Mev)$	$\frac{\mu}{m}$	$(\alpha_g)_{BS}$	$(\alpha_g)_{Sch}$	$\left[1 - \frac{(\alpha_g)_{Sch}}{(\alpha_g)_{BS}}\right] \cdot 100$
139	0.148	0.135	0.119	12 %
600	0.640	0.525	0.396	25 %

Conclusion

- The relativistic covariant amplitudes
  $$\mathcal{A} = \mathcal{A}^{J_1 J_2}(p; y, y'), \qquad \mathcal{A} = I + g^2 K \mathcal{A}$$
  where K is a symmetric kernel which is Hermitian for $p^2 < 4m^2$.
- $K = K_0 + K_I \Longrightarrow$
  $$\begin{cases} \mathrm{Tr}\ K_0^2 = \infty & \text{"fall at center" \& continuous spectrum} \\ \mathrm{Tr}\ K_I^2 < \infty & \text{bound states} \end{cases}$$
- $U = g^2 [K_0 + K_I] U \ \Longrightarrow\ \Phi = g^2 \mathcal{K}_I \Phi,$
  $\mathcal{K}_I = \frac{1}{\sqrt{1 - g^2 K_0}} K_I \frac{1}{\sqrt{1 - g^2 K_0}}, \qquad \mathrm{Tr}\ \mathcal{K}_I^2 < \infty.$
- $1 = g^2 \max\limits_U \frac{(U K_I U)}{(U[1 - g^2 K_0] U)} \qquad g^2 < g_c^2,$
- $1 = g^2 \max\limits_U \frac{(U K_I U)}{(UU)}, \qquad ||K_0|| \ll ||K_I||$
- Relativistic corrections are important.