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New properties of conformal transformations for scalar and Dirac particles in Riemannian and Riemann-Cartan spacetimes

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OUTLINE

- **Covariant Klein-Gordon equation and conformal invariance for massless particles.**
Fifty-year history
- **Conformal invariance and new (Hermitian) form of the Klein-Gordon equation. Conformal symmetry for a pointlike scalar particle (Higgs boson)**
- **Conformal symmetries of Hamiltonians**
- **Exact Foldy-Wouthyusen transformation**
- **Inclusion of electromagnetic interactions**
- **Comparison of scalar and Dirac particles**
- **Summary**



**Covariant Klein-Gordon equation
and conformal invariance for
massless particles. *Fifty-year
history***

Conformal invariance for a massless particle



R. Penrose



**N.A. Chernikov
(1928 – 2007)**



E. Tagirov

R. Penrose, In: Relativity, Groups and Topology.

London: Gordon and Breach, **1964**, p. 565.

N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincarè **9**, 109 (1968).

Covariant Klein-Gordon equation with the nonminimal coupling:

$$\left(\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} + m^2 - \lambda R \right) \psi = 0, \quad \lambda = \frac{1}{6}.$$

Non-minimal coupling with the scalar curvature

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R^{\lambda}_{\mu\lambda\nu}.$$

*The sign of the Penrose-Chernikov-Tagirov term depends
on the definition of R*

Republication of: Conformal treatment of infinity

Roger Penrose

The utility of this idea rests on the fact that the zero rest-mass free-field equations for each spin value are conformally invariant if interpreted suitably. For example, for spin zero, if the wave equation is written as

$$\left\{ \nabla_{\mu} \nabla^{\mu} + \frac{R}{6} \right\} \phi = 0$$

where R is the scalar curvature and ∇_{μ} denotes covariant derivative—both according to the metric $g_{\mu\nu}$ of \mathcal{M} , then

$$\left\{ \tilde{\nabla}_{\mu} \tilde{\nabla}^{\mu} + \frac{\tilde{R}}{6} \right\} \tilde{\phi} = 0$$

where $\tilde{\nabla}_{\mu}$, \tilde{R} refer to the metric $\tilde{g}_{\mu\nu} = \Omega^{-2}g_{\mu\nu}$ of $\tilde{\mathcal{M}}$ and where
 $\tilde{\phi} = \Omega\phi$.

N. Chernikov and E. Tagirov, Ann. Inst. Henri Poincaré **9**, 109 (1968)

$$\square - \frac{n-2}{4(n-1)} R = \Omega^{\frac{n+2}{2}} \left(\square' - \frac{n-2}{4(n-1)} R' \right) \Omega^{\frac{2-n}{2}},$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi, \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}.$$

n is a number of dimensions

But the only discovered pointlike scalar particle (Higgs boson) is massive!

Chernikov and Tagirov discussed an importance of non-minimal coupling with the scalar curvature for massive particles

New step ahead (Hamiltonian approach):

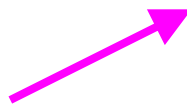
A. Accioly and H. Blas, Exact Foldy-Wouthuysen transformation for real spin-0 particle in curved space, PHYSICAL REVIEW D **66**, 067501 (2002).

Static metric in isotropic coordinates:

$$ds^2 = V(\mathbf{x})^2 (dx^0)^2 - W(\mathbf{x})^2 (d\mathbf{x})^2.$$

Feshbach-Villars transformation (useless for massless particles):

$$\psi = \phi + \chi, \quad \frac{i}{m} \frac{\partial \psi}{\partial t} = \phi - \chi.$$



the mass in the denominator!

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{V^2}{2W^3} \Delta W + \frac{V}{2W^2} \Delta V - \frac{1}{6} V^2 R,}$$

$$-\frac{1}{6} V^2 R = \frac{1}{6} F \Delta F - \frac{V^2}{2W^3} \Delta W - \frac{V}{2W^2} \Delta V,$$

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{1}{6} F \Delta F}, \quad F = \frac{V}{W}.$$

$$\text{If } m = 0, \quad \mathcal{H}_{FW} \left((g')^{\mu\nu} = \frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2} \right) = \mathcal{H}_{FW} (g^{\mu\nu}).$$

$$ds^2 = V^2 (dx^0)^2 - W^2 (d\mathbf{x})^2.$$

All terms except for the first term are conformally invariant
 Conformal transformation changes only such terms in the Foldy-
 Wouthuysen Hamiltonian which are proportional to the particle mass

But it is only a shadow of the conformal invariance,
 because $m \neq 0!$

Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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(Received 29 May 2013; published 5 August 2013)


- 1. Exact Foldy-Wouthuysen transformation for**
 - i) general static metric**
 - ii) frame rotating in the Kerr field approximated by a spatially isotropic metric**
- 2. Foldy-Wouthuysen Hamiltonian is conformally invariant for massless particles and conformally symmetric for massive ones**
- 3. Proof of similarity of conformal transformations for scalar and Dirac particles**

Scalar particle in general inertial and gravitational fields and conformal invariance revisited

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The results obtained in Ref. [4] and in the present study allow us to state the general property of conformal symmetry for massive particles. Conformal transformation (2) changes only such terms in the FW Hamiltonian which are proportional to the particle mass m . This property is valid not only for real scalars (Higgs boson) but also for compound ones (zero-spin atoms and nuclei).



**Conformal invariance and new
(Hermitian) form of the Klein-
Gordon equation. Conformal
symmetry for a pointlike scalar
particle
(Higgs boson)**

$$\left(\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} + m^2 - \lambda R \right) \psi = 0.$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi, \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}.$$

The nonunitary transformation

$$\Phi = \sqrt{g^{00}} \sqrt{-g} \psi, \quad g = \det g_{\mu\nu}, \quad g' = \Omega^{2n} g.$$

Φ is *invariant* relative to conformal transformations

We multiply the KG equation from left by

Hermitian form of the KG equation:

$$\left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} + \frac{m^2}{g^{00}} \right) \Phi = 0.$$

For a massless particle

$$\left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} \right)$$
$$= \left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} \right) .$$

This part of the KG equation can be transformed:

Denotations:

$$f = \sqrt{g^{00}} \sqrt{-g}, \quad \Gamma^i = \sqrt{-g} g^{0i},$$

$$G^{ij} = g^{ij} - \frac{g^{0i} g^{0j}}{g^{00}}, \quad \Upsilon = \frac{1}{2f} \left\{ \partial_i, \Gamma^i \right\} \frac{1}{f} = \frac{1}{2} \left\{ \partial_i, \frac{g^{0i}}{g^{00}} \right\},$$

$$\Lambda = -\frac{f_{,0,0}}{f} - \left(\frac{g^{0i}}{g^{00}} \right)_{,i} \frac{f_{,0}}{f} - 2 \frac{g^{0i}}{g^{00}} \frac{f_{,0,i}}{f} - \left(\frac{g^{0i}}{g^{00}} \right)_{,0} \frac{f_{,i}}{f}$$

$$- \frac{1}{2} \left(\frac{g^{0i}}{g^{00}} \right)_{,0,i} - \frac{1}{2f^2} \left(\frac{g^{0i}}{g^{00}} \right)_{,i} \Gamma^j_{,j} - \frac{g^{0i}}{2f^2 g^{00}} \Gamma^j_{,i,j}$$

$$+ \frac{1}{4f^4} \left(\Gamma^i_{,i} \right)^2 - \left(\frac{G^{ij}}{g^{00}} \right)_{,i} \frac{f_{,j}}{f} - \frac{G^{ij}}{g^{00}} \frac{f_{,i,j}}{f} - \frac{\lambda R}{g^{00}}.$$

Equivalent Hermitian form of the KG equation:

$$\left[(\partial_0 + \Upsilon)^2 + \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \Lambda + \frac{m^2}{g^{00}} \right] \Phi = 0.$$

Υ , G^{ij} / g^{00} , and Λ are invariant relative to conformal transformations

We can extend conformal symmetry on massive particles. The conformal-like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m$$

conserves the operator [...] acting on Φ and, therefore, conserves Φ when $\lambda=1/6$ (generally, $\lambda=(n-2)/[4(n-1)]$). As a result, the operator acting on Φ in the equation

$$\left(\frac{1}{\sqrt{g^{00}} \sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} \frac{1}{\sqrt{g^{00}} \sqrt{-g}} - \frac{\lambda R}{g^{00}} + \frac{m^2}{g^{00}} \right) \Phi = 0$$

remains unchanged. The initial covariant Klein-Gordon equation

$$\left(\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu} + m^2 - \frac{n-2}{4(n-1)} R \right) \psi = 0$$

has following properties:

$$\square + m^2 - \frac{n-2}{4(n-1)} R$$

$$= \Omega^{\frac{n+2}{2}} \left(\square' + m'^2 - \frac{n-2}{4(n-1)} R' \right) \Omega^{\frac{2-n}{2}},$$

$$\psi' = \Omega^{\frac{n-2}{2}} \psi \quad \text{when} \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

The conformal symmetry has extended on a pointlike scalar particle (Higgs boson)!



Conformal symmetries of Hamiltonians

Feshbach-Villars transformation to a Hamiltonian form for a *massive* particle

$$\left[(\partial_0 + \Upsilon)^2 + \mathcal{T} \right] \Phi = 0, \quad \mathcal{T} = \partial_i \frac{G^{ij}}{g^{00}} \partial_j + \Lambda + \frac{m^2}{g^{00}}$$

$$\Phi = \phi + \chi, \quad i(\partial_0 + \Upsilon)\Phi = m(\phi - \chi).$$

***m* appears in a denominator!**

Generalized Feshbach-Villars transformation for both massive and massless particles

The wave function in the Feshbach-Villars representation is given by

$$\Psi = \frac{1}{2} \begin{pmatrix} \Phi + \frac{i}{m} (\partial_0 + \Upsilon) \Phi \\ \Phi - \frac{i}{m} (\partial_0 + \Upsilon) \Phi \end{pmatrix}.$$

It was proved that a similar transformation can be performed with the use of any nonzero parameter instead of the particle mass, ***m***

Successive generalized Feshbach-Villars and Foldy-Wouthyusen transformations

The method has been developed in

A.J. Silenko, Hamilton operator and the semiclassical limit for scalar particles in an electromagnetic field, Theor. Math. Phys. **156**, 1308 (2008).

$$\psi = \phi + \chi, \quad \frac{i}{N}(\partial_0 + \Upsilon)\psi = \phi - \chi. \quad N \text{ is an arbitrary nonzero real parameter}$$

$$i(\partial_0 + \Upsilon)\Psi = \left[\frac{N}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \frac{T}{2N} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right] \Psi, \quad \Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

The Pauli matrices ρ_i can be used

$$i(\partial_0 + \Upsilon)\Psi = \frac{1}{2N} \left[\rho_3 (N^2 + T) + i\rho_2 (-N^2 + T) \right] \Psi.$$

Therefore, we obtain the following generalized Feshbach-Villars Hamiltonian:

$$\mathcal{H}_{gFV} = \rho_3 \frac{N^2 + T}{2N} + i\rho_2 \frac{-N^2 + T}{2N} - iY.$$

This Hamiltonian *is not changed* by the conformal-like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1} m.$$

General method of the Foldy-Wouthyusen transformation

$$\mathcal{H}_{gFV} = \rho_3 \mathcal{M} + \mathcal{E} + \mathcal{O}, \quad \rho_3 \mathcal{M} = \mathcal{M} \rho_3, \quad \rho_3 \mathcal{E} = \mathcal{E} \rho_3, \quad \rho_3 \mathcal{O} = -\mathcal{O} \rho_3,$$

$$\mathcal{M} = \frac{N^2 + T}{2N}, \quad \mathcal{E} = -i\Upsilon, \quad \mathcal{O} = i\rho_2 \frac{-N^2 + T}{2N},$$
$$U = \frac{\varepsilon + \mathcal{M} + \rho_3 \mathcal{O}}{\sqrt{2\varepsilon(\varepsilon + \mathcal{M})}}, \quad \varepsilon = \sqrt{\mathcal{M}^2 + \mathcal{O}^2}, \quad \mathcal{F} = \mathcal{E} - i \frac{\partial}{\partial t}.$$

In the case under consideration,

$$U = \frac{\varepsilon + N + \rho_1(\varepsilon - N)}{2\sqrt{\varepsilon N}}, \quad \varepsilon = \sqrt{T}, \quad \mathcal{H}' = \rho_3 \varepsilon + \mathcal{E}' + \mathcal{O}',$$
$$\mathcal{E}' = \mathcal{E} + \frac{1}{2\sqrt{\varepsilon}} \left[\sqrt{\varepsilon}, \left[\sqrt{\varepsilon}, \mathcal{F} \right] \right] \frac{1}{\sqrt{\varepsilon}}, \quad \mathcal{O}' = \rho_1 \frac{1}{2\sqrt{\varepsilon}} \left[\varepsilon, \mathcal{F} \right] \frac{1}{\sqrt{\varepsilon}}.$$

The transformed (intermediate) Hamiltonian describing the both massive and massless particles does not contain N and is not changed by the conformal-like transformation!

Final approximate Foldy-Wouthuysen Hamiltonian

$$\mathcal{H}_{FW} = \rho_3 \varepsilon + \mathcal{E}', \quad \varepsilon = \sqrt{T}.$$

The conformal-like transformation does not change the Foldy-Wouthuysen Hamiltonian!

Conformal symmetries of the generalized Feshbach-Villars and Foldy-Wouthuysen Hamiltonians have been proved
in the general form!



Exact Foldy-Wouthyusen transformation

Sufficient condition of exact Foldy-Wouthuysen transformation

$$[\mathcal{M}, \mathcal{O}] = [\mathcal{F}, \mathcal{O}] = 0 \quad \Rightarrow \quad \mathcal{H}_{FW} = \rho_3 \sqrt{T} - i\Upsilon.$$

1. Foldy-Wouthuysen transformation is exact for *any static metric*

In many important cases, the spacetime metric can be represented in a static form with an appropriate coordinate transformation. For example, it can be made for de Sitter and anti-de Sitter spaces)

Static metric in isotropic coordinates

Exact Foldy-Wouthuysen transformation has been performed by Accioly and H. Blas (2002) for massive particles

$$ds^2 = V(\mathbf{x})^2 (dx^0)^2 - W(\mathbf{x})^2 (d\mathbf{x})^2.$$

The obtained result formally coincides with that by
Accioly and Blas

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{V^2}{2W^3} \Delta W + \frac{V}{2W^2} \Delta V - \frac{1}{6} V^2 R,}$$

$$-\frac{1}{6} V^2 R = \frac{1}{6} F \Delta F - \frac{V^2}{2W^3} \Delta W - \frac{V}{2W^2} \Delta V,$$

$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{1}{6} F \Delta F}, \quad F = \frac{V}{W}.$$

But we already have a right to consider the case of $m=0$
showing the conformal invariance!

$$\text{If } m=0, \quad \mathcal{H}_{FW} \left((g')^{\mu\nu} = \frac{g^{\mu\nu}}{\Omega(\mathbf{x})^2} \right) = \mathcal{H}_{FW} (g^{\mu\nu}).$$

2. Frame rotating in the Kerr field approximated by a spatially isotropic metric

All effects of the Schwarzschild gravitational field and the frame rotation are *exactly* described in *isotropic* Arnowitt-Deser-Misner coordinates

Effects of rotation of the *Kerr* source are described in these coordinates within terms of order of $O(a^2 r^{-4})$

total mass M , total angular momentum $J=Mca$

This case covers an observer on the ground of the Earth or on a satellite. It reproduces not only the well-known effects of the rotating frame but also the Lense-Thirring effect.

Frame rotating in the Kerr field: An approximation by a spatially isotropic metric

$$d\mathbf{x}' = d\mathbf{x} - [\boldsymbol{\Omega}(r) \times \mathbf{x}] dx^0, \quad \boldsymbol{\Omega}(r) = \boldsymbol{\omega}(r) - \mathbf{o}, \quad \mathbf{o} = \text{const},$$

$$ds^2 = V^2(r) (dx^0)^2 - W^2(r) (d\mathbf{x} - \mathbf{K} dx^0) (d\mathbf{x} - \mathbf{K} dx^0),$$

$$\mathbf{K} = \boldsymbol{\Omega} \times \mathbf{x},$$

$$g^{00} = \frac{1}{V^2(r)}, \quad g^{0i} = \frac{K^i}{V^2(r)}, \quad G^{ij} = -\frac{\delta^{ij}}{W^2(r)}.$$

In isotropic spherical coordinates ($\boldsymbol{\Omega} = \Omega \mathbf{e}_z$),

$$ds^2 = \left[V^2(r) - W^2(r) \Omega^2(r) r^2 \sin^2 \theta \right] (dx^0)^2$$

$$- W^2(r) \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2 + 2\Omega(r) \sin^2 \theta dx^0 d\phi) \right].$$

Exact Foldy-Wouthyusen Hamiltonian

$$T = m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{V}{2W^2} \left[F \left(\frac{2W'_r}{r} + W''_{rr} \right) + \frac{2V'_r}{r} + V''_{rr} \right] - \frac{1}{6} V^2 R,$$

$$-\frac{1}{6} V^2 R = \frac{1}{6} F \Delta F - \frac{V}{2W^2} \left[F \left(\frac{2W'_r}{r} + W''_{rr} \right) + \frac{2V'_r}{r} + V''_{rr} \right] + \frac{1}{12} \Omega_r'^2 r^2 \sin^2 \theta,$$


$$\mathcal{H}_{FW} = \rho_3 \sqrt{m^2 V^2 + F \mathbf{p}^2 F - \frac{1}{4} \nabla F \cdot \nabla F + \frac{1}{6} F \Delta F + \frac{1}{12} \Omega_r'^2 r^2 \sin^2 \theta + \mathbf{\Omega} \cdot \mathbf{l}},$$

$$\mathbf{\Omega} = -\mathbf{0} \quad \text{for rotating frame,} \quad \mathbf{\Omega} = \frac{2G\mathbf{J}}{r^3} \quad \text{for Lense-Thirring metric,}$$

$$\mathbf{\Omega} = \frac{2G\mathbf{J}}{r^3} \left[1 - \frac{3GM}{r} + \frac{21G^2 M^2}{4r^2} + \mathcal{O}\left(\frac{a^2}{r^2}\right) \right] \quad \text{for approximate Kerr metric,}$$

$\mathbf{l} = \mathbf{r} \times \mathbf{p}$ is operator of angular momentum.

The Hamiltonian *is not changed* by the conformal-like transformation



Inclusion of electromagnetic interactions

**Electromagnetic interactions can be added
as follows:**

$$\left[g^{\mu\nu} (\nabla_{\mu} + ieA_{\mu}) (\nabla_{\nu} + ieA_{\nu}) + m^2 - \lambda R \right] \psi = 0 .$$

Simple derivation leads to

$$\left[\frac{1}{\sqrt{-g}} (\partial_{\mu} + ieA_{\mu}) \sqrt{-g} g^{\mu\nu} (\partial_{\nu} + ieA_{\nu}) + m^2 - \lambda R \right] \psi = 0 .$$

**Equivalent (relative to conformal-like
transformations) form of this equation is given by**

$$\left[(D_0 + \Upsilon')^2 + D_i \frac{G^{ij}}{g^{00}} D_j + \Lambda + \frac{m^2}{g^{00}} \right] \Phi = 0,$$

Λ is the same

$$\Upsilon' = \frac{1}{2} \left\{ D_i, \frac{g^{0i}}{g^{00}} \right\}, \quad D_{\mu} = \partial_{\mu} + ieA_{\mu}.$$

Generalized Feshbach-Villars transformation for both massive and massless particles

$$\left[(\partial_0 + \Upsilon')^2 + T' \right] \Phi = 0, \quad T' = D_i \frac{G^{ij}}{g^{00}} D_j + \Lambda + \frac{m^2}{g^{00}},$$

$$\Phi = \phi + \chi, \quad i(D_0 + \Upsilon')\Phi = N(\phi - \chi).$$

The nonunitary transformation results in the following generalized Feshbach-Villars Hamiltonian:

$$\mathcal{H}_{g_{FV}} = \rho_3 \frac{N^2 + T'}{2N} + i\rho_2 \frac{-N^2 + T'}{2N} - i\Upsilon'.$$

This Hamiltonian *is not changed* by the conformal-like transformation $g'_{\mu\nu} = \Omega^2 g_{\mu\nu}$, $m' = \Omega^{-1} m$.

The considered equations *do not change* their conformal properties when electromagnetic interactions are included



Comparison of scalar and Dirac particles

Conformal transformations for Dirac particles

gravity + electromagnetism

Covariant Dirac equation:

$$\left(i\hbar\gamma^a D_a - mc\right)\psi = 0, \quad D_a = e_a^\mu \partial_\mu + \frac{i}{4}\sigma^{bc}\Gamma_{bca}.$$

General nonunitary transformation brings this equation to the Hermitian Hamiltonian form:

$$\Phi = \sqrt{\sqrt{-g}e_{\hat{0}}^0} \psi.$$

Yu.N. Obukhov, A.J. Silenko, and O.V. Teryaev, Phys. Rev. D **84**, 024025 (2011).

The Hamiltonian *is not changed* by the conformal-like transformation

$$g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1}m.$$

Dirac and Foldy-Wouthuysen Hamiltonians are conformally invariant when $m=0$!

Wave function of the initial covariant Dirac equation possesses the following conformal property:

$$\psi' = \Omega^{3/2}\psi \quad \text{when} \quad g'_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad m' = \Omega^{-1}m.$$

Squared Dirac equation:

$$\left[\pi^i \pi_i - \frac{\hbar}{2} \sigma^{\alpha\beta} \left(\frac{q}{c} F_{\alpha\beta} + m \Phi_{\alpha\beta} \right) + \frac{\hbar^2}{4} R \right. \\ \left. + \frac{\hbar^2}{16} (2\Gamma^i_{\alpha\beta} \Gamma_i^{\alpha\beta} + i \varepsilon^{\alpha\beta\mu\nu} \Gamma^i_{\alpha\beta} \Gamma_{i\mu\nu} \gamma_5) - m^2 c^2 \right] \psi = 0,$$

← electromagnetic field tensor

For the squared Dirac equation, $\lambda = 1/4!$

Summary

- The covariant Klein-Gordon equation is presented in a new (Hermitian) form and conformal symmetry for a massive pointlike scalar particle (**Higgs boson**) is found. Conformal transformations for a massless particle and conformal-like transformations for a massive one do not change the form of the obtained equation
- Generalized Feshbach-Villars transformation and Foldy-Wouthyusen transformation are performed for both massive and massless scalar particles in arbitrary gravitational fields. Conformal symmetries of relativistic Hamiltonians are found in the general case.
- Exact Foldy-Wouthyusen transformations are fulfilled for an arbitrary static metric and for a frame rotating in the Kerr field approximated by a spatially isotropic metric
- It is proven that inclusion of electromagnetic interactions does not change conformal properties of the considered equations
- Conformal symmetries of equations for scalar and Dirac particles are very similar



Thank you for your attention