The integral equations solution for quantum two particles systems with the Cornell potential in momentum space

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Annotation

A new high precision method for solution of integral equations in the momentum space with Cornell potential is suggested. The method can be used effectively for the bound state equations.

Outline

Motivation (1)

At the last XII-th School-Seminar (2013) we told about obtaining the kernel of the radial equation of a quark–antiquark relativistic system with an arbitrary angular momentum with the use of the amplitude of the one-gluon exchange and requirement of quark confinement in the context of the Poincare-covariant model based on relativistic Hamiltonian dynamics (RHD).

In the general case, the wave function (WF) of a bound system of spinor quarks with masses m_q, m_Q and respectively with 4-momentums p_1, p_2 and helicities $\lambda_{1,2}$ in RHD satisfies the three-dimensional integral equation [a]:

$$\sum_{\lambda_1,\lambda_2} \int < \mathbf{k}', \sigma_1, \sigma_2 \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 > \Phi^{J\mu}_{\lambda_1\lambda_2}(\mathbf{k}')\, \mathrm{d}\mathbf{k}' =$$

$$= \left(M - \sqrt{\mathbf{k}^2 + m_q^2} - \sqrt{\mathbf{k}^2 + m_Q^2}\right) \Phi^{J\mu}_{\sigma_1\sigma_2}(\mathbf{k}) \qquad (1)$$

[a] *Keister B.D., Polyzou W.N. Adv. Nucl. Phys. v.20. 1991.*

The knowledge of the kernel of the RHD integral equation makes it possible to switch to the procedure of numerical solution and calculation of the spectrum of masses of the quarkonium.

Motivation (2)

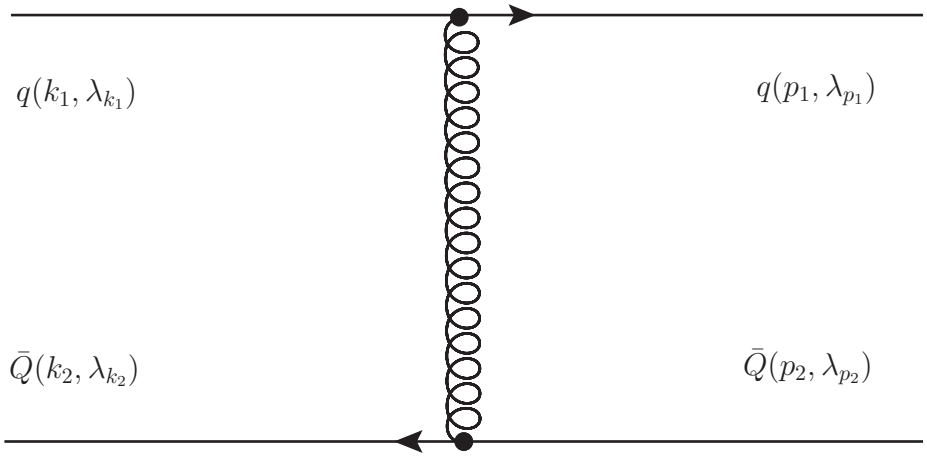
The interaction potential can be constructed using the amplitude of the elastic scattering of particles that construct the system [a].

$$q_i(k_1, \lambda_{k_1}) + \bar{q}_j(k_2, \lambda_{k_2}) \to q_k(p_1, \lambda_{p_1}) + \bar{q}_l(p_2, \lambda_{p_2}) \ ,$$

where particle momenta and spin numbers are given between parentheses and $i, j, k, l = 1, 2, 3$ – color degrees of freedom.

[a] *Lucha et. al Phys Rev D44 1991*

In the first non-zero order of the perturbation theory the main contribution to the amplitude of elastic scattering T_{fi} of quark-antiquark determined by the diagram of one-gluon exchange



One-gluon exchange contribution V_{pert}

As a result the potential generated by diagram of gluon exchange including the conservation of the currents and the running coupling constant of QCD will take the form:

$$V^{(\mathrm{pert})}_{\lambda_{p_1},\lambda_{p_2}\ \lambda_{k_1},\lambda_{k_2}}\left(\mathbf{k}',\mathbf{k}\right)=(-1)\frac{N_{k,k'}}{8\pi^2}\frac{4\alpha_s\left(q^2\right)}{3q^2}J^{\mu}\left(1\right)\widetilde{D}_{\mu\nu}\left(q\right)J^{\nu}\left(2\right)=$$

$$=(-1)\frac{N_{k,k'}}{8\pi^2}\frac{4\alpha_s\left(q^2\right)}{3q^2}\left[j^{\mu}\left(1\right)j_{\mu}\left(2\right)-\frac{q_{\mu}j^{\mu}\left(1\right)q_{\nu}j^{\nu}\left(2\right)}{q^2}\right]\ . \tag{2}$$

In non-relativistic limit $\left(\mathrm{k}^2,\mathrm{k}'^{\,2}\ll m_q^2,m_Q^2\right)$ potential (2) in coordinate space takes the form of the Coulomb potential [a]

$$\hat{V}_C\left(\mathrm{r}\right)=-\frac{4\alpha_s}{3\mathrm{r}}\ . \tag{3}$$

Therefore we can assume (2) as the relativistic generalization of the Coulomb potential (3).

[a] *Lucha, Schoberl: hep-ph/9601263.*

The confining component of the interquark potential V_{conf}

The confining component of the interquark potential can be derived by analyzing the Lorentz structure of the potential and experimental data on the meson mass spectrum.

The analysis results that the nonperturbative part of the interquark potential is determined as the sum of the vector ($\sim K_V(q^2)$) and scalar ($\sim K_S(q^2)$) confining parts [a]:

$$
\begin{aligned}
&< \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V}_{conf} \parallel \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} >= \\
&= \frac{N_{k,k'}}{(2\pi)^3} \left[K_V(q^2) \bar{u}_{\lambda_{p_1}}(p_1) \left(\gamma^\mu + \frac{\mathrm{i}\kappa_q}{2m_q} (k_1 - p_1)_\nu \sigma^{\mu\nu} \right) u_{\lambda_{k_1}}(k_1) \times \right. \\
&\times \bar{v}_{\lambda_{k_2}}(k_2) \left(\gamma^\mu + \frac{\mathrm{i}\kappa_Q}{2m_Q} (p_2 - k_2)_\nu \sigma^{\mu\nu} \right) v_{\lambda_{p_2}}(p_2) + \\
&\left. + K_S(q^2) \bar{u}_{\lambda_{p_1}}(p_1) u_{\lambda_{k_1}}(k_1) \bar{v}_{\lambda_{k_2}}(k_2) v_{\lambda_{p_2}}(p_2) \right] , \qquad (4)
\end{aligned}
$$

where functions

$$
K_V\left(q^2\right) = \frac{8\pi A_V}{\mathbf{q}^4} + \delta\left(\mathbf{k}' - \mathbf{k}\right) B_V\left(\mathrm{k}\right) , \quad K_S\left(q^2\right) = \frac{8\pi A_S}{\mathbf{q}^4} + \delta\left(\mathbf{k}' - \mathbf{k}\right) B_S\left(\mathrm{k}\right) . \qquad (5)
$$

should provide confinement of the quarks in the meson.

[a] *Galkin, Mishurov, Faustov // Sov. J. Nucl. Phys. – 1992. – Vol. 55. – P. 1207.*

The confining component of the interquark potential V_{conf} (2)

Expression (4) for potential give in non-relativistic limit the linear confining potential $V(\mathrm{r}) = \sigma \mathrm{r} + w$ with parameters

$$\sigma = (A_V - A_S), \quad w = (B_V - B_S) \ . \tag{6}$$

Thus we can say that sum

$$< \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V}_{pert} + \hat{V}_{conf} \parallel \ \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} > \tag{7}$$

defined by relations (2) and (4) is relativistic generalization of the Cornell potential (8).

But in structure of (7) we meet terms which give singularities in initial equation. At this place we have strong reasons for good numerical method to stay making calculations in momentum space unlike to do different expansion by velocities and so on.

Absolute the same type of hypersingular integrals we meet in bound state problems with the Cornell potential in momentum space. Further, for simplicity, we demonstrate how to calculate numerically integrals of this type with high precision on the example of non-relativistic Schrodinger equation with the Cornell potential.

The Cornell potential

In the description of the meson, as a system of a quark and an antiquark is widely used the Cornell potential. Effective centrally symmetric potential interaction between quarks with constituent masses m_q and m_Q contains Coulomb and linear part:

$$\hat{V}\left(\mathrm{r}\right) = -\frac{4\alpha_s}{3\mathrm{r}} + \sigma\mathrm{r} + w \ , \quad \mathrm{r} = |\mathbf{r}| \ , \tag{8}$$

where σ , w – model parameters, α_s – QCD strong coupling constant.
This potential satisfies the requirement of quark confinement and has been widely used in calculations of the spectra of heavy mesons [[a], [b]].

[a] *Eichten, Kinoshita, Gottfried // Phys.Rev.Lett. – 1975. – 34. – P. 369.*
[b] *Eichten, Kinoshita, Gottfried // Phys.Rev. – 1980. – D21. – P. 203.*

Singular integrals

After transformation of integral $[0,\infty) \to [-1,1]$ by making substitution $k = c\frac{1+t}{1-t}$ in

$$\frac{k^2}{2\mu}\phi_l(k) + \int_0^\infty V_l(k,k')\phi_l(k')k'^2 dk' = E\phi_l(k)\ , \tag{9}$$

singular integrals appear [a].

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[a] *Deloff A. // Ann.Phys.(N.Y.) – 2006. – 34. – P. 369.*

for Coulomb part

$$V^C(r) = -\frac{\alpha}{r} \quad \Rightarrow \quad V_l^C(k,k') = -\frac{\alpha}{\pi}\frac{Q_l(y)}{kk'} \quad \Rightarrow \quad \int_{-1}^{1} f(t) ln|t-z| dt \quad |z| \le 1 \tag{10}$$

for linear part

$$V^L(r) = \frac{r}{a^2} \quad \Rightarrow \quad V_l^L(k,k') = \frac{Q_l'(y)}{\pi(akk')^2} \quad \Rightarrow \quad \int_{-1}^{1} \frac{f(t)}{(t-z)^2} dt \quad |z| \le 1, \tag{11}$$

where

$$y = \frac{k^2 + k'^2}{2kk'}$$

Alternative methods

In paper [a] it was shown that "power of singularity" in integral (11) can be "reduced" by making integration by parts but instead of integral equation we receive integral-differential equation.

$$\int_{-1}^{1} \frac{f(t)}{(t-z)^2} dt \quad |z| \leq 1, \qquad \Rightarrow \qquad \int_{-1}^{1} \frac{f(t)}{(t-z)} dt \quad |z| \leq 1 \tag{12}$$

[a] *Deloff A. // Ann.Phys.(N.Y.) – 2006. – 34. – P. 369.*

This method has relatively low precision $\left(\sim 10^{-6}\right)$.

In the same paper [a] was suggested **Semispectral Chebyshev method** for integrals (14) and (13):

$$\int_{-1}^{1} f(t) ln|t-z| dt \quad |z| \leq 1 \tag{13}$$

Improved semi-spectral Chebyshev method (1)

In paper [a] was shown that using property $\int\limits_0^\infty Q_0'(y)\,dk' = 0$ we can introduce contour-term and rewrite terms $\sim Q_0'(y)$ in a form

$$\int\limits_0^\infty Q_0'(y)\,\phi_0(k')dk' \quad \Rightarrow \quad \int\limits_0^\infty Q_0'(y)\left(\phi_0(k') - \phi_0(k)\right)dk' \tag{14}$$

and substitution of pure expression

$$Q_0'(y) = \frac{1}{1-y^2} = -\left(\frac{2kk'}{k'+k}\right)^2 \frac{1}{(k'-k)^2}\,. \tag{15}$$

leads (in the case of Non-rel. Schodinger eq. with Cornell potential) to integral like

$$\frac{4\sigma}{\pi}\int\limits_0^\infty \frac{P_l(y)}{(k'+k)^2}\,\frac{[\phi_\ell(k') - \phi_\ell(k)]}{(k-k')^2}\,dk'; \tag{16}$$

[a] *Tang A. MKN theory of bound states – e-Print: hep-ph/0103035v4*

Improved semi-spectral Chebyshev method (2)

After some calculations we have received new quadrature formula for numerical calculation of such type singular integrals which combine advantages of paper [a] and Lande subtraction method

$$\int\limits_{-1}^{1} \frac{\phi_\ell(t)-\phi_\ell(z)}{(t-z)^2}\,dt = \sum_{j=1}^{N} \omega_j(z)\phi_\ell(t_j)\ . \tag{17}$$

$$\omega_j(z) = \frac{2}{N} {\sum_{i=1}^{N}}' T_{i-1}(t_j)\ X_{i-1}(z)\ , \tag{18}$$

$$X_n(z) = 2 {\sum_{k=0}^{n-1}}' U_{n-1-k}(z) \left\{ T_k(z) \ln\left|\frac{1-z}{1+z}\right| + R_k(z) \right\}\ , \tag{19}$$

$$R_n(z) = 2 {\sum_{k=0}^{n-1}}' T_k(z) \left[\frac{(-1)^{(n-k)+1}+1}{(n-k)} \right]\ . \tag{20}$$

[a] *Deloff A. // Ann.Phys.(N.Y.) – 2006. – 34. – P. 369.*

At the same time the type of initial equation remains to be integral as before in contrast to paper [a] and accuracy of numerical calculations increases!

Checking for integrals with exact solution (1)

Sample 1: lets select function $f_1(t) = t^4$

$$I_1(f_1) = \int_{-1}^{1} \frac{t^4 - z^4}{(t-z)^2} dt = \frac{2}{3} + 6z^2 + 4z^3 ln \left| \frac{1-z}{1+z} \right|, |z| < 1 \tag{21}$$

Table: Sample 1: Numerical test.

z	Exact value, by (21)	Quadrature formula (17)	δ, %
-0.99	-13.997086845834733	-13.997086845834787	3.807×10^{-15}
-0.7	1.2267940186741848	1.2267940186741804	3.61992×10^{-15}
-0.3	1.1398104321587945	1.1398104321587932	1.16885×10^{-15}
-0.1	0.7258639838848181	0.7258639838848198	2.29428×10^{-15}
0.0	0.6666666666666666	0.6666666666666636	4.4964×10^{-15}
0.1	0.7258639838848181	0.7258639838848202	2.90609×10^{-15}
0.5	1.617360522332611 7	1.617360522332613 7	1.23559×10^{-15}
0.7	1.2267940186741848	1.2267940186741875	2.17195×10^{-15}
0.99	-13.997086845834733	-13.997086845834758	1.77673×10^{-15}

Checking for integrals with exact solution (2)

Sample 2: lets select function $f_2(t) = e^{-t}$

$$I_2(f_2) = \int_{-1}^{1} \frac{e^{-t} - e^{-z}}{(t-z)^2} dt \tag{22}$$

$$I_2(f_2) = \frac{e^{-1-z}}{z^2-1}\{e(z^2-1)[Ei(z+1) - Ei(z-1)] - e^{z+2}(z-1) + e^z(z+1) - 2e\}\ , |z| < 1 \tag{23}$$

where $Ei(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} dt$ exponential integral.

Table: Sample 2: Numerical test.

z	Exact value, by (23)	Quadrature formula (17)	δ, %
-0.99	-18.597791752712833	-18.597791752712908	4.0116×10^{-15}
-0.6	-0.9833492587254643	-0.9833492587253971	6.83058×10^{-14}
-0.2	0.6793649549585903	0.6793649549585903	5.5563×10^{-15}
0.0	1.0283404811209695	1.0283404811209635	5.82998×10^{-15}
0.1	1.1445832768980895	1.1445832768980928	2.90994×10^{-15}
0.3	1.3046004784874934	1.3046004784874938	3.40402×10^{-16}
0.7	1.518977651147536	1.518977651147532	2.63125×10^{-15}
0.99	3.3470563271679046	3.3470563271677065	5.91755×10^{-14}

Non-relativistic Schrodinger eq. with linear potential for the case $l = 0$

As a second numerical test we have done calculations of non-relativistic Schrodinger equation with linear potential for the case $l = 0$ for which the exact solutions are well known [a] Energy can be found by formula $\epsilon(0, s, 0) = -s^{2/3} z_\nu$, where $s \equiv 1/2\mu a$, and $z_\nu \quad (\nu = 1, 2, 3, \ldots)$ – zeros of Airy function $Ai(z)$. The results of this calculation is presented in Table1. There is very good agreement between two results with excellent precision.

[a] *Abramowitz, M. Handbook of Mathematical Functions / M.Abramowitz, I.Stegun // Dover, New York – 1972.*

Table: Comparison of results of calculations for linear potential in momentum space by using our quadrature formula with exact value received by Airy function zeros. Number of points $N = 100$.

n	E_n (17)	E_n (exact value)	δ, %
1	2.3381074104597843	2.338107410459767	7.33×10^{-13}
2	4.0879494441309765	4.087949444130971	1.33×10^{-13}
3	5.520559828095326	5.520559828095551	4.06×10^{-12}
4	6.786708090071581	6.78670809007176	2.62×10^{-12}
5	7.944133587120411	7.944133587120854	5.58×10^{-12}
6	9.022650853340487	9.022650853340982	5.50×10^{-12}
7	10.040174341556877	10.040174341558087	1.21×10^{-11}

In momentum space there is no results with accuracy better than $10^{-5} \div 10^{-6}$.

Comparison results in momentum space with results of in coordinate space

Like a crosscheck we also made comparison our results for solution of (9) with Cornell potential in momentum space for the first eigenvalue of energy E_1 for different values of α - coefficient, with results of calculation in coordinate space by authors in [a].

[a] *Kang, D. Precise numerical solutions of potential problems using Crank-Nicholson method / D.Kang, E.Won // e-Print arXiv: physics/060917v1 [physics.comp-ph] (Dated: February 2, 2008)*

We should noted that our method let us repeat numbers from [a] with very good precisions already with $N = 100$ points, while that last was received on mesh $N = 300000$.

Table: Comparison of the results of $1S$-state energy calculations for Cornell potential in momentum space by using our quadrature formula with results for coordinate space (ζ_1) in [a].

α	$E_1(1S), N = 100$	$\zeta_1(1S)$ [a], $N = 300000$	$\Delta\zeta$
0.0	2.338 107 410 459 784 3	2.338 107 410 458 750	1.0×10^{-12}
0.2	2.167 316 208 772 692 5	2.167 316 208 771 731	1.0×10^{-12}
0.4	1.988 503 899 750 148 7	1.988 503 899 749 943	9.6×10^{-13}
0.6	1.801 073 805 647 306	1.801 073 805 646 145	8.5×10^{-13}
0.8	1.604 408 543 236 034 9	1.604 408 543 235 973	6.6×10^{-13}
1.0	1.397 875 641 659 084	1.397 875 641 659 578	3.8×10^{-13}
1.2	1.180 833 939 742 701	1.180 833 939 744 863	2.1×10^{-14}
1.4	0.952 640 495 217 967 7	0.952 640 495 219 193	5.8×10^{-13}

Conclusions

1) We suggest for actively using the new high precision quadrature formula for singular integrals like in equations with the Cornell potential.

2) Numerical test of quadrature formula in case of problems with the Cornell potential is performed and shown the good accuracy of method.

3) Next steps » now we create "instrument" which let us carry out numerical calculation in momentum space with high precision not only for the problems of description quarkonium states but also for the simplest electrodynamic systems.

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Thanks for your attention !!!

Backup Slides

Eigenvalue problem

$$\sum_{i=1}^{N}\left(W_{ji}+\frac{\tilde{k}_j^2}{2\mu a}\delta_{ji}-\epsilon\right)\phi_i(\tilde{k})=0 \tag{24}$$

$$\sum_{i=1}^{N}\left(\frac{\tilde{k}_j^2}{2\mu a}\delta_{ji}+W_{ji}^{LOG}+W_{ji}\right)\phi_i(\tilde{k})=\epsilon\phi_i(\tilde{k}) \tag{25}$$

Description of a bound systems in RHD (1)

In the RHD -based Poincare-covariant quark model, the "creation" of a relativistic bound system is begun with constructing a two-particle system of quarks with 4-momentums

$$p_1 = \left(\omega_{m_q}(\mathrm{p}_1)\,, \mathbf{p}_1\right)\ , p_2 = \left(\omega_{m_Q}(\mathrm{p}_2)\,, \mathbf{p}_2\right)$$

and masses m_q and m_Q, which is described with the use of unitary representations of the Poincare group.

Then an interaction $\hat{V}$ is introduced so that the requirement of Poincare invariance be satisfied for the system of interacting particles. This requirement is realized in the form of a Poincare algebra on the manifold of observable dynamic systems.

To separate the relative motion and the motion of the center of inertia in the Poincare-covariant model, the quark momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ can be transformed to a total momentum $\mathbf{P}$ and a relative momentum $\mathbf{k}$:

$$\mathbf{P} = \mathbf{p_1} + \mathbf{p_2}\ , \tag{26}$$

$$\mathbf{k} = \frac{1}{2}\left(\mathbf{p}_1 - \mathbf{p}_2\right) + \frac{\mathbf{P}}{\widetilde{M}_0\left(\omega_{\widetilde{M}_0}(\mathrm{P}) + \widetilde{M}_0\right)}\left(m_Q^2 - m_q^2 - \widetilde{M}_0\left[\omega_{m_Q}(\mathrm{p}_2) - \omega_{m_q}(\mathrm{p}_1)\right]\right) \tag{27}$$

where introduced an effective mass of two quarks:

$$\widetilde{M}_0 = \sqrt{\left[\omega_{m_Q}(\mathrm{p}_2) + \omega_{m_q}(\mathrm{p}_1)\right]^2 - \mathbf{P}^2}$$

Description of a quarkonium in RHD (2)

In the general case, the wave function (WF) of a bound system of spinor quarks with masses m_q, m_Q and respectively with 4-momentums p_1, p_2 and helicities $\lambda_{1,2}$ in RHD satisfies the three-dimensional integral equation [a]:

$$\sum_{\lambda_1,\lambda_2} \int < \mathbf{k}', \sigma_1, \sigma_2 \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 > \Phi^{J\mu}_{\lambda_1\lambda_2}\left(\mathbf{k}'\right) \mathrm{d}\mathbf{k}' =$$

$$= \left(M - \sqrt{\mathbf{k}^2 + m_q^2} - \sqrt{\mathbf{k}^2 + m_Q^2}\right) \Phi^{J\mu}_{\sigma_1\sigma_2}\left(\mathbf{k}\right) \qquad (28)$$

with reduced matrix element:

$$< \mathbf{P}', \mathbf{k}', \sigma_1, \sigma_2 \left| \hat{V} \right| \mathbf{P}, \mathbf{k}, \lambda_1, \lambda_2 >= \delta\left(\mathbf{P}' - \mathbf{P}\right) < \mathbf{k}', \sigma_1, \sigma_2 \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 >$$

where $\hat{V}$ – interaction potential, M – mass of system, J, μ – total angular momentum and its projection.

[a] *Keister B.D., Polyzou W.N. Adv. Nucl. Phys. v.20. 1991.*

Description of a quarkonium in RHD (3)

For deriving the one-dimensional radial equation we use the Clebsch–Gordan expansion of the Poincare group. Since it is convenient to use the spiral states of quarks for analytic calculation of the kernel, and the bound states are classified using the orbital moment L, total spin S, and total angular momentum J, we shall perform the calculation of the kernel in two stages.

At the first stage, we construct, in the system of the center of inertia $\mathbf{P} = 0$, a state with quantum numbers J and μ and with quark helicities λ_1 and λ_2, which form the basis of a nonreducible two-particle space of the Poincare group $|\mathbf{P} = 0, J, \mu, k, \lambda_1, \lambda_2 >$.

At the second stage, with the help of the Clebsch–Gordan coefficients of the group of rotations, $C^{1/2\ \ 1/2\ \ S}_{\lambda_1\ -\lambda_2\ \ \lambda}$ and $C^{L\ \ S\ \ J}_{0\ \ \lambda\ \ \lambda}$ we obtain a basis of states with quantum numbers J, μ, L, and S:

$$| \mathrm{k}, J, \mu, L, S >= \sum_{\lambda_1, \lambda_2} \sqrt{\frac{2L+1}{2J+1}}\ C^{1/2\ \ 1/2\ \ S}_{\lambda_1\ -\lambda_2\ \ \lambda}\ C^{L\ \ S\ \ J}_{0\ \ \lambda\ \ \lambda}\ | \mathrm{k}, J, \mu, \lambda_1, \lambda_2 >$$

Description of a quarkonium in RHD (4)

The radial equation for a two-particle bound state in the center-momentum system ($\mathbf{P}=0$) after separation of variables has the form [1]:

$$\sum_{\ell',S'}\int_0^\infty V^J_{\ell',S';\ell,S}\left(\mathrm{k}',\mathrm{k}\right)\Phi^{J\mu}_{\ell',S'}\left(\mathrm{k}'\right)\mathrm{k}'^2\mathrm{dk}'=\left(M-\sqrt{\mathbf{k}^2+m_q^2}-\sqrt{\mathbf{k}^2+m_Q^2}\right)\Phi^{J\mu}_{\ell,S}\left(\mathrm{k}\right) \quad (29)$$

where the operator $V^J_{\ell',S',\ell,S}\left(\mathrm{k}',\mathrm{k}\right)=<\mathrm{k}',J,\mu,\ell',S'\parallel\hat{V}\parallel\mathrm{k},J,\mu,\ell,S>$ of the potential of the quark–antiquark system is determined by the relation

$$V^J_{\ell',S';\ell,S}\left(\mathrm{k}',\mathrm{k}\right)=\frac{\sqrt{\left(2L+1\right)\left(2L'+1\right)}}{2J+1}\sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'}C^{1/2\ 1/2\ S}_{\lambda_1\ -\lambda_2\ \lambda}C^{L\ S\ J}_{0\ \lambda\ \lambda}$$

$$C^{1/2\ 1/2\ S'}_{\lambda_1'\ -\lambda_2'\ \lambda'}C^{L'\ S'\ J}_{0\ \lambda'\ \lambda'}<\mathrm{k}',J,\mu,\lambda_1',\lambda_2'\parallel\hat{V}\parallel\mathrm{k},J,\mu,\lambda_1,\lambda_2> \quad (30)$$

[1] *Keister B.D., Polyzou W.N. Adv. Nucl. Phys. v.20. 1991.*

Description of a quarkonium in RHD (5)

Matrix element $< \mathrm{k}', J', \mu', \lambda_1', \lambda_2' \parallel \hat{V} \parallel \mathrm{k}, J, \mu, \lambda_1, \lambda_2 >$ deals with $< \mathbf{k}', \lambda_1', \lambda_2' \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 >$ by Jacob-Wick's expansion (see, e.g. [a]) and after usage of Wigner-Eckart theorem leads to simple relation:

$$V^J_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}\left(\mathrm{k}', \mathrm{k}\right) = \int\limits_{-1}^{1} \mathrm{d}\left(\cos\beta\right) \int\limits_{0}^{2\pi} \mathrm{d}\varphi \; D^J_{\lambda,\lambda'}\left(\varphi, \beta, -\varphi\right) \left\langle \mathbf{k}', \lambda_1', \lambda_2' \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 \right\rangle \tag{31}$$

where

$\cos\beta = (\mathbf{k}\mathbf{k}') \,/\, (|\mathbf{k}|\,|\mathbf{k}'|)$,

$D^J_{\mu\ \lambda}\left(\varphi_k, \theta_k, -\varphi_k\right)$ – function which gives matrices of the irreducible representation of $SU\left(2\right)$ group by index J. The explicit form of the D matrix determined in term of spherical angles of relative momentum vector (27) $\mathbf{k} = \{\sin\theta_k \cos\varphi_k, \sin\theta_k \sin\varphi_k, \cos\theta_k\}$,

$\lambda = \lambda_1/2 - \lambda_2/2$, $\lambda' = \lambda_1'/2 - \lambda_2'/2$ – combinations of doubled value of quarks helicities $\lambda_{1,2} = \pm 1$.

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[a] *G.E. Brown and A.D. Jackson 1976*

So, our goal is to construct the relativistic QCD–motivated kernel of the radial equation of quarkonium (30) for arbitrary total angular momentum J.

Calculation of the spinor part of the potential (8)

The spinor part of equations (2) can be calculated by the method of basis spinors [a]. In this approach, the fermion chain with an operator Q , which is expressed as a combination of Dirac matrices, can be represented in the form

$$\bar{u}_{\lambda_p}(p,s_p)\,Q\,u_{\lambda_k}(k,s_k) = \sum_{\sigma,\rho=-1}^{1}\sum_{A,C=-1}^{1} \bar{u}_{\lambda_p}(p,s_p)\,u_{-\sigma}(b_{-C})\times$$
$$\{\bar{u}_{\sigma}(b_C)\,Q\,u_{-\rho}(b_{-A})\}\,\bar{u}_{\rho}(b_A)\,u_{\lambda_k}(k,s_k)$$
$$= \sum_{\sigma,\rho=-1}^{1}\sum_{A,C=-1}^{1} \bar{s}^{(C,1)}_{\lambda_p,\sigma}(p)\,\Gamma^{C,A}_{\sigma,\rho}[Q]\,s^{(A,1)}_{\rho,\lambda_k}(k)\ ,(\sigma,\rho,C,A=\pm 1)\ , \qquad (32)$$

where the coefficients of the expansion on basis spinors $s^{(A,B)}_{\rho,\lambda_k}(k)$ for the spiral states are defined by the relations ($B=1$ corresponds to fermion, and $B=-1$ for antifermion):

$$s^{(A,B)}_{\rho,\lambda_k}(k) = f^{B}_{\lambda,\rho} D^{*\,1/2}_{A\ \rho/2,-B\lambda_k/2}(\varphi_k,\theta_k,\varphi_k)\,W_{m_k}(-B\rho\lambda_k\times \mathrm{k})\ , \qquad (33)$$

$$f^{B}_{\lambda,\rho} = \delta_{B,1}\lambda + \delta_{B,-1}\rho \qquad (34)$$

with the auxiliary function

$$W_{m_k}(\pm\mathrm{k}) = \sqrt{\omega_{m_k}(\mathrm{k})\pm \mathrm{k}}\ ,\quad k^2 = \omega^2_{m_k}(\mathrm{k}) - |\mathbf{k}|^2 = m_k^2\ . \qquad (35)$$

[a] *Andreev // Phys.Atom.Nucl. – 2003. – Vol. 66. – P. 383.*

One-gluon exchange contribution V_{pert} (2)

Using the Feynman rules it's easy to find the corresponding scattering amplitude. The one-gluon exchange contribution V_{pert} to the kernel of quark interaction in centre-momentum system is given by the expression:

$$< \mathbf{k}', \lambda_{p_1}, \lambda_{p_2} \parallel \hat{V}_{\rm pert} \parallel \mathbf{k}, \lambda_{k_1}, \lambda_{k_2} > \equiv V^{(\rm pert)}_{\lambda_{p_1},\lambda_{p_2}\ \lambda_{k_1},\lambda_{k_2}} \left(\mathbf{k}', \mathbf{k}\right) =$$

$$= (-1) \frac{N_{k,k'}}{8\pi^2} \frac{4\alpha_s}{3q^2} j^{\mu}\left(1\right) \widetilde{D}_{\mu\nu}\left(q\right) j^{\nu}\left(2\right) \ , \qquad (36)$$

with currents

$$j^{\mu}\left(1\right) = \bar{u}_{\lambda_{p_1}}\left(p_1\right) \gamma^{\mu} u_{\lambda_{k_1}}\left(k_1\right) \ , \quad j^{\mu}\left(2\right) = \bar{v}_{\lambda_{k_2}}\left(k_2\right) \gamma^{\mu} v_{\lambda_{p_2}}\left(p_2\right) \ ,$$

where $q = \{0, \mathbf{k} - \mathbf{k}'\}$ – transfer momentum,

$N_{k,k'} = 1/\sqrt{\omega_{m_q}\left(\mathrm{k}\right) \omega_{m_Q}\left(\mathrm{k}\right) \omega_{m_q}\left(\mathrm{k}'\right) \omega_{m_Q}\left(\mathrm{k}'\right)}$
and

$$k_1 = \left(\omega_{m_q}\left(\mathrm{k}\right), \ \mathbf{k}\right), \ p_1 = \left(\omega_{m_q}\left(\mathrm{k}'\right), \ \mathbf{k}'\right),$$

$$k_2 = \left(\omega_{m_Q}\left(\mathrm{k}\right), \ -\mathbf{k}\right), \ p_2 = \left(\omega_{m_Q}\left(\mathrm{k}'\right), \ -\mathbf{k}'\right) \ .$$

One-gluon exchange contribution V_{pert} (3)

Function $\widetilde{D}_{\mu\nu}(q)$, deals with gluon propagator

$$\widetilde{D}_{\mu\nu}\left(q\right)=\left(g_{\mu\nu}-\frac{q_{\mu}q_{\nu}}{q^{2}}\right) \tag{37}$$

In contrast to elastic scattering in the c.m.s. for construction of potential (36) we should take into account virtuality states of quarks, which leads to $|\mathbf{k}| \neq |\mathbf{k}'|$ [a].

[a] *Пилькун, Х. Физика релятивистских частиц / — Москва: Мир , 1983. — 542 с.*

One-gluon exchange contribution V_{pert} (4)

Conservation of current in the quark-gluon vertex requires [a]

$$q_\mu j^\mu (1) = q_\mu j^\mu (2) = 0 \, . \tag{38}$$

[a] *Keister B.D., Polyzou W.N. Adv. Nucl. Phys. v.20. 1991.*

To perform requirement of gauge invariance of (38) we should modify currents $j^\mu (1,2)$, in the next form [a]:

$$j^\mu (1,2) \to J^\mu (1,2) = \left(g^\mu_\nu - \frac{q^\mu q_\nu}{q^2} \right) j^\nu (1,2) \, . \tag{39}$$

[a] *Klink, W. H. // Few Body Syst. — 2003. — Vol. 33. — P. 99–110.*

One-gluon exchange contribution V_{pert} (6)

The calculation of $V^{J}_{\lambda_1',\lambda_2';\lambda_1,\lambda_2}(\mathrm{k}',\mathrm{k})$ which determine the perturbative part of the kernel of the integral equation (29), will hold two stages:

1 – calculation the spinor part of the potential $\langle \mathbf{k}', \lambda_1', \lambda_2' \parallel \hat{V} \parallel \mathbf{k}, \lambda_1, \lambda_2 \rangle$,

2 – integration over the angle variables

Procedure of finding the interquark potential (7)

After calculation of the spin part by using (31) and the trivial integration over the azimuthal angle φ, we obtain the kernel of relativistic fermion-fermion system in the "L-S" basis for any total angular momentum J, (full spin moment $S = 0, 1$):

$$V^{J}_{L',S';L,S}\left(\mathrm{k}',\mathrm{k}\right) = \frac{\sqrt{\left(2L+1\right)\left(2L'+1\right)}}{2J+1} \times$$

$$\times \sum_{\lambda_1,\lambda_2,\lambda_1',\lambda_2'} C^{1/2\ \ 1/2\ \ S}_{\lambda_1/2\ -\lambda_2/2\ \lambda/2}\, C^{L\ \ S\ \ J}_{0\ \lambda/2\ \lambda/2} C^{1/2\ \ 1/2\ \ S'}_{\lambda_1'/2\ -\lambda_2'/2\ \lambda'/2}\, C^{L'\ S'\ \ J}_{0\ \lambda'/2\ \ \lambda'/2} \times$$

$$\times \left(V^{\mathrm{I}}_{\lambda_{k_1},\lambda_{k_2},\lambda_{p_1},\lambda_{p_2}} + V^{\mathrm{II}}_{\lambda_{k_1},\lambda_{k_2},\lambda_{p_1},\lambda_{p_2}} + \right.$$

$$\left. + V^{\mathrm{III}}_{\lambda_{k_1},\lambda_{k_2},\lambda_{p_1},\lambda_{p_2}} + V^{\mathrm{IV}}_{\lambda_{k_1},\lambda_{k_2},\lambda_{p_1},\lambda_{p_2}} + V^{(\mathrm{V})}_{\lambda_{p_1},\lambda_{p_2}\ \lambda_{k_1},\lambda_{k_2}} \right) . \qquad (40)$$

Procedure of finding the interquark potential (3)

As a result, the kernel of radial equation defined by the first part of potential (2) after a trivial integration over the angle φ can be written as

$$V^{(\mathrm{I})}_{\lambda_{p_1},\lambda_{p_2},\lambda_{k_1},\lambda_{k_2}}\left(\mathrm{k}',\mathrm{k}\right)=2\sum_{\sigma,\rho=-1}^{1}W_{-\sigma\lambda_{k_1},\rho\lambda_{k_2}}\left(\mathrm{k}\right)\,W_{-\sigma\lambda_{p_1},\rho\,\lambda_{p_2}}\left(\mathrm{k}'\right)\Big[\delta_{\lambda_{k_1},\lambda_{k_2}}\rho\,\sigma\,\times$$

$$\times G^{J,1/2,1/2}_{-\lambda_{k_1},\lambda_{k_1};\lambda_{p_1},\lambda_{p_2}}\left[\tilde{\mathrm{R}}_\ell\left(\mathrm{k}',\mathrm{k}\right)\right]+\delta_{\sigma\lambda_{k_2},-\rho\lambda_{k_1}}\,G^{J,1/2,1/2}_{\lambda_{k_1},\lambda_{k_2};\lambda_{p_1},\lambda_{p_2}}\left[\tilde{\mathrm{R}}_\ell\left(\mathrm{k}',\mathrm{k}\right)\right]\Big]\,. \qquad (41)$$

Procedure of finding the interquark potential (4)

Similar calculations for the second part of the potential (2)

$$V^{(\mathrm{II})}_{\lambda_{p_1},\lambda_{p_2}\ \lambda_{k_1},\lambda_{k_2}}\left(\mathbf{k}',\mathbf{k}\right)=\frac{N_{k,k'}}{8\pi^2}\frac{4\alpha_s\left(q^2\right)}{3q^4}\left(qj_{\lambda_{p_1},\lambda_{k_1}}\left(p_1,k_1\right)\right)\left(qj_{\lambda_{p_2},\lambda_{k_2}}\left(p_2,k_2\right)\right) \tag{42}$$

lead to a contribution in the form

$$V^{(\mathrm{II})}_{\lambda_{p_1},\lambda_{p_2},\lambda_{k_1},\lambda_{k_2}}\left(\mathrm{k}',\mathrm{k}\right)= \tag{43}$$

$$G^{J,1/2,1/2}_{\lambda_{k_1},\lambda_{k_2};\lambda_{p_1},\lambda_{p_2}}\left[\tilde{U}\left(\mathrm{k}',\mathrm{k}\right)\right]\sum_{\sigma,\rho=-1}^{1}W_{-\sigma\lambda_{k_1},\rho\lambda_{k_2}}\left(\mathrm{k}\right)W_{-\sigma\lambda_{p_1},\rho\ \lambda_{p_2}}\left(\mathrm{k}'\right)\ .$$

Procedure of finding the interquark potential (6)

Functions $\tilde{R}_\ell\left(\mathrm{k}',\mathrm{k}\right)$ and $\tilde{U}_\ell\left(\mathrm{k}',\mathrm{k}\right)$, included in the equation (41) and (44), expressed as one-dimensional integrals:

$$\tilde{R}_\ell\left(\mathrm{k}',\mathrm{k}\right)=-\frac{1}{3\pi}\int\limits_{-1}^{1}\frac{\alpha_s\left(q^2\right)\mathrm{P}_\ell\left(x\right)}{q^2}\,\mathrm{d}x\,, \tag{44}$$

$$\tilde{U}_\ell\left(\mathrm{k}',\mathrm{k}\right)=\varrho\left(\mathrm{k}',\mathrm{k}\right)\int\limits_{-1}^{1}\frac{\alpha_s\left(q^2\right)\mathrm{P}_\ell\left(x\right)}{q^4}\,\mathrm{d}x\,. \tag{45}$$

Factor $\varrho_{12}\left(\mathrm{k}',\mathrm{k}\right)$ в (45)

$$\varrho\left(\mathrm{k}',\mathrm{k}\right)=\left(\omega_{m_q}\left(\mathrm{k}'\right)-\omega_{m_q}\left(\mathrm{k}\right)\right)\left(\omega_{m_Q}\left(\mathrm{k}\right)-\omega_{m_Q}\left(\mathrm{k}'\right)\right) \tag{46}$$

makes the dimension of functions $\tilde{R}_\ell$ and $\tilde{U}_\ell$ the same.